



A relativity-based framework for statistical testing guided by the independence of ancillary statistics: methodology and nonparametric illustrations

Albert Vexler¹ · Douglas Landsittel¹

Received: 3 November 2025 / Accepted: 16 March 2026
© The Author(s) 2026

Abstract

A test statistic is typically constructed to discriminate effectively between competing hypotheses. In contrast, we propose and examine a framework that shifts attention to ancillary statistics—quantities whose distributions remain invariant under the tested hypotheses. Rather than directly optimizing discriminatory power, the proposed approach seeks to construct test statistics that exhibit relative independence from ancillary structures. We show that reducing the dependence between a test statistic and a vector of ancillary statistics can yield the most powerful (MP) decision-making procedure. We establish a Basu-type independence result and show that certain forms of MP test statistics characterize the underlying distribution. These principles are developed through decision-theoretic arguments and illustrated in two nonparametric applications. Ancillary-guided modifications of the Shapiro–Wilk, Anderson–Darling, Cramér–von Mises, and Kolmogorov–Smirnov tests deliver twofold efficiency gains under symmetric alternatives. In multivariate mean testing, a simple trace-normalized statistic reduces ancillary dependence and then outperforms Hotelling’s procedure under heavy-tailed distributions, while the classical test remains optimal under normality. The proposed framework is simple to implement and provides a theoretically grounded strategy for enhancing the power of statistical tests in practice.

Keywords Ancillary statistics · Characterization · Basu’s theorem · Most powerful tests · Multivariate mean testing · Nonparametric tests · Goodness-of-fit

1 Introduction

The development of decision-making mechanisms lies at the heart of statistical theory and underpins a wide range of inferential procedures in practice. This paper explores

✉ Albert Vexler
avexler@buffalo.edu

Douglas Landsittel
dplansit@buffalo.edu

¹ Department of Biostatistics, The State University of New York at Buffalo, Buffalo, NY 14214, USA

a strategy for constructing, improving, and comparing test statistics, emphasizing performance gains achieved through structural modifications guided by the principle of independence from ancillary statistics.

To illustrate the ideas of this paper, consider a simple example. Suppose we observe a sample of n independent and identically distributed (i.i.d.) data points X_1, \dots, X_n with variance 1, and our goal is to test the hypothesis $H_0 : E(X_1) = 0$ versus $H_1 : E(X_1) > 0$. A natural choice of test statistic is based on the sample mean $\bar{X}_n = \sum_{i=1}^n X_i/n$, which discriminates between the hypotheses via the rule: We reject H_0 if $n^{0.5} \bar{X}_n > C$, where C is a test threshold. The law of large numbers and the central limit theorem justify this choice by ensuring that \bar{X}_n converges to its expectation and has asymptotically distinct distributions under H_0 and H_1 . This reflects a classical strategy: selecting a statistic with strong discriminatory power.

Alternatively, within a relativity-based framework, one may focus on constructing or modifying test statistics to be (exactly or approximately) independent of certain ancillary quantities—statistics whose distributions are invariant (or approximately invariant) under the null and alternative. For instance, continuing with the example above, the sample variance, $S_n^2 = \sum_{i=1}^n (X_i - \bar{X}_n)^2/n$, is ancillary in this context. By seeking a test statistic that is independent of S_n^2 , one can develop or refine procedures that exhibit improved performance. More broadly, this idea can guide both the construction of new test statistics and the modification of existing ones, by reducing their dependence on a relevant set of ancillary statistics. For example, modifying the sample mean to reduce its dependence on the sample variance can yield a more powerful test when the distribution of X_1 is skewed (Vexler and Hutson 2024). It is worth noting that when X_1 follows a normal distribution, the statistic \bar{X}_n is independent of the sample variance S_n^2 and can represent a most powerful test statistic in the Neyman–Pearson sense.

We use the term “relativity” in the following sense. A standard route to powerful decisions is to seek statistics with relevant maximal discriminating ability. Our results show that when the conditions identified in this paper are satisfied, an apparently opposite approach—constructing a system of statistics that is “powerless”—can become relatively the same as a maximal-discrimination-based rule. Informally, creating a best-discriminating statistic and developing a system of powerless statistics may be viewed as two equivalent representations of the same inferential content. The connection to relativity theory is analogical: The ancillary system can be viewed as a reference frame, and once inference is expressed relative to this frame, the resulting conclusion is stable across different but equivalent constructions.

That is, by identifying a collection of statistics with no discriminatory ability—denote this set by A —one can construct a function of the data that is independent of A , and thereby obtain a statistic that is highly efficient for testing the hypotheses of interest.

In this context, an intriguing question arises: How should the set A be chosen to ensure maximal testing efficiency? This leads to several related inquiries. For example, can we define A so that a statistic independent of A yields a most powerful test? In practice, if several ancillary statistics A_1, \dots, A_k are available, which among them should be used to guide the transformation of a test statistic? Should A be selected to characterize the data distributions under the null and alternative hypotheses, or chosen

as a maximal ancillary statistic in some formal sense? These questions are examined in this research, but many relevant aspects remain open for further investigation in future research.

Recently, Vexler and Hutson (2024) showed that transforming a test statistic T to achieve independence from a relevant ancillary statistic A can yield a new statistic T_N with greater power than the original T . In this paper, we associate the term “Most Powerful (MP) Test” with ancillarity by defining a class of relevant ancillary statistics such that independence between a test statistic and this class plays a central role in achieving efficient test performance.

We adopt a theoretical formalism in which, under the considered decision-making structure, an MP mechanism exists—for example, the Neyman–Pearson likelihood ratio test. In many practical settings, however, an MP test may be unavailable. In such cases, we interpret the term “MP” as a theoretical ideal: a target that can be closely approached by approximating the underlying assumptions and structural principles discussed in the methodological sections of this paper. As an application, we consider nonparametric tests for normality that exemplify this perspective.

Remark 1 (*Relation to conditioning on ancillaries, invariance, and robustness*) A common theme in the ancillary-statistics literature is to use ancillarity primarily for *calibration*: one conditions on (exact or approximate) ancillary information to remove nuisance-driven variability and obtain sharper conditional likelihood/pivot approximations for parametric inference, yielding a conditional analysis rather than a general recipe for modifying an existing test statistic (Fraser 2004; Ghosh et al. 2010).

A closely related reduction principle in hypothesis testing is *invariance*: When a transformation group leaves the testing problem unchanged, one restricts attention to group-invariant decision rules (equivalently, functions of a maximal invariant), which leads to exact randomization-based tests under the induced transformation distribution; recent work shows that power can depend materially on how the acting (sub)group is chosen (Koning and Hemerik 2024).

Robust testing pursues a different objective—stability of level and power under outliers, heavy tails, or mild misspecification—typically by robustifying score/Wald/likelihood constructions or by using divergence-based criteria (Héritier and Ronchetti 1994; Cantoni and Ronchetti 2001; Toma and Broniatowski 2011).

The present paper takes a complementary, *test-construction* perspective: We treat ancillary structure not mainly as something to condition on (calibration) or to quotient out by symmetry (invariance), but as a design constraint—build or adjust a test so that the decision rule is as insensitive as possible to ancillary variation. This viewpoint is extended to include a characterization of the MP ideal (MP test procedures), providing a conceptual benchmark based on independence from ancillary structures and identifying conditions under which such a characterization holds. This perspective leads to concrete construction and modification rules that reduce “ancillary leakage”—so that ancillary variation has minimal impact on the rejection decision—thereby concentrating power on features that distinguish the competing hypotheses in the regimes targeted by the paper. This, for example, creates an opportunity to present nonparametric applications of the proposed technique.

Ancillarity underpins many foundational statistical concepts (Lehmann and Scholz 1992; Ghosh et al. 2010). Perhaps one of the most widely recognized results in the study of ancillary statistics is Basu's theorem (Ghosh 2002). In simple terms, Basu's theorem states that if one has a complete sufficient statistic—containing all the information about an unknown parameter—and an ancillary statistic—containing no information about the parameter—then these two statistics are independent.

It appears that we are particularly interested in a type of conversion of Basu's result. However, it is important to note that a direct or naïve reversal of the classical Basu theorem—e.g., concluding sufficiency from independence—is generally false (Kagan and Malinovsky 2013).

This paper deals with an extension of Basu's theorem and its conversion-type results to decision-making fields. We propose and examine the relativity-based framework, showing relevant examples. A consequence of this perspective is characterizations of the normal and negative exponential distributions. In analogy to these results, we refer the reader to Pfanzagl (1968) for an example of characterizing the one-parameter exponential family through the existence of MP test statistics.

We emphasize the practical value of the proposed methodology in goodness-of-fit and multivariate mean testing. In nonparametric settings, the main results developed in this study may be applied approximately, in much the same spirit as the Neyman–Pearson framework offers guidance for constructing test statistics beyond strict parametric assumptions. In both applications, the method is simple, transparent, and efficient.

Remark 2 (*Approximate MP benchmark and ancillary independence*) Throughout this paper, we distinguish between exact independence (as in the formal results of Sects. 2 and 3) and approximate independence, which is what is typically achievable in nonparametric settings or when nuisance components must be estimated. We also distinguish exact ancillarity from approximate ancillarity (e.g., Skovgaard 1985), where a statistic has approximately the same distribution under the null and alternative hypotheses. By “approximate independence,” we mean that the dependence between the proposed test statistic and the (exactly or approximately) ancillary component is small (as can be assessed by a dependence measure) or vanishes in an appropriate asymptotic regime under both hypotheses. In such cases, the MP property is used only as a conceptual benchmark: Reducing ancillary dependence is a design constraint intended to mitigate ancillary leakage and improve power, but it does not by itself guarantee exact optimality. Accordingly, when only approximate ancillarity or approximate independence is available, the MP conclusions of Sects. 2 and 3 are used only as a guiding ideal leading to power improvements, not as a claim of exact optimality.

The remainder of the paper is organized as follows. Section 2 introduces the necessary notation and preliminaries. Section 3 develops a decision-theoretic approach that motivates the construction of test statistics with reduced dependence on ancillary structures and establishes conditions under which such procedures are MP. Moreover, a result in the spirit of Basu's theorem is reformulated and proved within a decision-theoretic context. The theoretical implications are illustrated through Examples 4–6. We also demonstrate that certain forms of MP test statistics implicitly characterize the

underlying data distribution. Section 4 presents nonparametric applications, including modifications of classical tests for normality guided by ancillary-independence principles and a multivariate mean testing example in which Hotelling’s procedure is compared with a simple trace-normalized test. Section 5 concludes with a discussion of methodological implications and potential directions for future research. Proofs of the main results are given in the Appendix, and additional technical derivations, along with several auxiliary remarks, are provided in the online Supplement.

2 Some notation and preliminaries

Without loss of generality and to simplify the exposition of the main objectives of this paper, we introduce the following formal notation.

Assume we observe data denoted by D , and our goal is to test a null hypothesis H_0 against an alternative H_1 . Let $T = T(D)$ be a real-valued test statistic constructed from D , such that the hypothesis H_0 is rejected when $T(D) > C$, for some fixed threshold C . We denote the probability and expectation under H_k by \Pr_k and E_k , respectively, for $k \in \{0, 1\}$.

Throughout this paper, we assume that the distributions \Pr_0 and \Pr_1 of D are absolutely continuous with respect to a σ -finite measure τ , defined over a class Υ of measurable subsets of a space \mathbb{X} , in which D takes values. Then, there exist nonnegative generalized density functions f_0 and f_1 with respect to τ such that for all $v \in \Upsilon$ and $k \in \{0, 1\}$, $\Pr_k(D \in v) = \int_v f_k(x) d\tau(x)$. Note that f_0 and f_1 need not belong to the same parametric family of distributions.

Define f_0^T and f_1^T as the probability density functions of the statistic $T(D)$ under H_0 and H_1 , respectively. Then, for each $k \in \{0, 1\}$ and any threshold $t \in \mathbb{R}^1$,

$$\Pr_k\{T(D) \leq t\} = \int_{\mathbb{X}} I\{T(x) \leq t\} f_k(x) d\tau(x) = \int_{-\infty}^t f_k^T(u) du,$$

where $I(\cdot)$ denotes the indicator function.

In this setting, we assume that there exists some most powerful (MP) test statistic, denoted by $\Lambda(D)$. For example, in many situations involving simple hypotheses, the Neyman–Pearson lemma implies that the likelihood ratio $\Lambda(D) = f_1(D)/f_0(D)$ defines the MP test statistic, assuming $f_0(x) > 0$ and $f_1(x) > 0$ for all $x \in \mathbb{X}$. In the multivariate case, f_0 and f_1 are interpreted as joint densities with respect to a dominating measure τ ; see Lehmann (1950).

A statistic $A = A(D)$ is said to be *ancillary* if its distribution is invariant under the hypotheses, i.e., $f_0^A(u) = f_1^A(u)$ for all u in the support of A . Thus, A cannot be used to discriminate between H_0 and H_1 .

Let $f_k^{T_1, T_2}(u, v)$ denote the joint density of two statistics $T_1(D)$ and $T_2(D)$ under H_k , and let $f_k^{T_1|T_2}(u, v)$ denote the conditional density of $T_1(D)$ given $T_2(D) = v$ under H_k . If T_1 and T_2 are independent under H_k , then, for all $u \in \text{supp}(T_1)$ and $v \in \text{supp}(T_2)$, $f_k^{T_1, T_2}(u, v) = f_k^{T_1}(u) f_k^{T_2}(v)$.

For conceptual simplicity, it will be assumed that in cases where a researcher plans to employ a test statistic $S(D) = L(\Lambda(D))$, where $L(u)$ is a strictly increasing function and $\Lambda(D)$ is MP, the test statistic in use is $T(D) = W(S(D))$, with $W = L^{-1}$ on the range of L . In such cases, we treat $T(D)$ as the MP test statistic.

We now restate a strengthened version of a result from Vexler and Hutson (2024), whose proof is revisited and streamlined in the present paper for completeness.

Let $A = A(D)$ be a statistic satisfying $f_1^A = f_0^A$; that is, A is ancillary. Suppose the test statistic $T = T(D)$ can be written as a function of two components, T_N and A , via $T = \psi(T_N, A)$, where ψ is a bivariate function. Assume further that T_N and A are independent under both H_0 and H_1 . The following result holds.

Proposition 1 *Let T_N be a statistic such that there exists a strictly increasing function $L(u) > 0$ for which $f_1^{L(T_N)}(u) = u f_0^{L(T_N)}(u)$, for all u . Then, the test that rejects H_0 for large values of T_N dominates the test based on T in terms of power.*

Proof See Appendix. □

This proposition illustrates that modifying a test statistic to be independent of an ancillary statistic can improve the power of the original test.

Remark 3 In connection with the condition $f_1^{L(T_N)}(u) = u f_0^{L(T_N)}(u)$, for all u , in Proposition 1, we observe the following. Assume we have a test statistic $Y = Y(D)$, and the likelihood ratio $L^Y(u) = f_1^Y(u)/f_0^Y(u)$ is a strictly increasing function with an inverse function $W(u)$. In this case, Y can be transformed to the form $Y_N = L^Y(Y)$, and

$$\begin{aligned} f_k^{Y_N}(u) &= \frac{d}{du} \Pr_k \{L^Y(Y) \leq u\} = \frac{d}{du} \Pr_k \{Y \leq W(u)\} \\ &= f_k^Y(W(u)) \frac{d}{du} W(u), \quad k \in \{0, 1\}. \end{aligned}$$

This implies

$0 < f_1^{Y_N}(u)/f_0^{Y_N}(u) = f_1^Y(W(u))/f_0^Y(W(u)) = L^Y(W(u)) = u$. Proposition 1 may alternatively be derived under the assumption that the identity $f_1^{L(T_N)}(u) = u f_0^{L(T_N)}(u)$ holds at $u = L(T_N)$ only (see the Appendix for details). For example, according to Hájek et al. (1967), for location parameter testing problems, $L^Y(Y)$ can be monotonic if and only if f_0^Y and f_1^Y are strongly unimodal [Hájek et al. 1967, pp. 32–33].

3 Converse-type results on ancillarity and test optimality: examples and characterizations

In this section, Proposition 2 addresses the natural question arising from Proposition 1: What constitutes an optimal choice of the ancillary statistic A to ensure that the modified test statistic becomes most powerful (MP). Proposition 3 presents a result analogous to Basu's theorem, reformulated and proven within a decision-theoretic

context. Proposition 4 further illustrates how the proposed theoretical principles can be used to characterize the distribution of the data. The proofs of the propositions are provided in the Appendix. Examples 4–6 involving tests for location and scale are provided to illustrate the theoretical framework. Additional insights and implications of the theoretical results are discussed in Remark 5 and in the Supplementary Material (see Remarks S1.1 and S1.2).

Assume that $V = V(D)$ denotes a set of ancillary statistics with a joint density satisfying $f_1^V(v) = f_0^V(v)$.

Proposition 2 *Let the mapping $D \mapsto (T_N, V)$ be injective with a measurable inverse. Assume T_N is a test statistic that is independent of V under both H_0 and H_1 , and that there exists a strictly monotonic function $L(u) > 0$ such that $f_1^{L(T_N)}(u) = u f_0^{L(T_N)}(u)$. Then, T_N is MP.*

Informally, Proposition 2 states that if a test statistic T_N is independent of a set of ancillary statistics V , and the pair (T_N, V) suffices to reconstruct the underlying data, then the test based on T_N is MP.

Remark 4 For vector-valued data D , a natural extension of Proposition 2 can be used as follows. Let T_N be conducted using a collection of statistics $M \in \mathbb{R}^p$, $p \geq 1$, such that the mapping $D \mapsto (M, V)$ is injective with a measurable inverse. Assume that M is independent of V under both H_0 and H_1 , and that there exists a strictly monotonic function $L(u) > 0$ satisfying $f_1^M(m) = L(T_N(m)) f_0^M(m)$, $m \in \mathbb{R}^p$. Then, the test based on T_N is MP. In this formulation, T_N is univariate, whereas M may be vector-valued.

Remark 5 Proposition 2 is plausibly related in spirit to Lemma 1 in Kagan (2002), though formulated, proved, and employed here in a decision-theoretic context.

An auxiliary result concerning data transformations and ancillary statistics is provided in the Supplementary Material (see Remark S1.1).

In the next simple examples, we illustrate how the conclusion of Proposition 2 may fail when one of its structural conditions is violated.

Example 1 (*Failure of T_N —ancillary independence $\Rightarrow T_N$ is not MP*) Let X_1, X_2 be i.i.d. $N(\mu, 1)$, and consider the simple one-sided problem: $H_0 : \mu = 0$ versus $H_1 : \mu = \delta$, $\delta > 0$.

Define the ancillary statistic $V = X_1 - X_2$, noting that $V \sim N(0, 2)$ under both H_0 and H_1 . Consider the test statistic $T_N = X_1$. The mapping $D = (X_1, X_2) \mapsto (T_N, V) = (X_1, X_1 - X_2)$ is injective with measurable inverse, since $X_2 = T_N - V$. However, T_N is not independent of V : $\text{Cov}(T_N, V) = 1 \neq 0$. It is clear that, for the one-sided alternative $\delta > 0$, the Neyman–Pearson (hence MP) test is based on $(X_1 + X_2)/2$. Thus, when the independence condition in Proposition 2 is violated, the statistic T_N need not yield an MP procedure.

Example 2 (*The pair (T_N, V) does not determine D , and T_N is not MP*) Let $D = (X_1, X_2)$, where X_1 and X_2 are independent. Consider the simple hypotheses: $H_0 :$

$X_1 \sim N(1, 1)$, $X_2 \sim N(-1, 1)$ versus $H_1 : X_1 \sim N(-1, 1)$, $X_2 \sim N(1, 1)$. Define $T_N = X_1 + X_2$, $V = |X_1 - X_2|$.

Note that V is ancillary in the two-regime sense, since, under H_0 , one has $X_1 - X_2 \sim N(2, 2)$, while under H_1 one has $X_1 - X_2 \sim N(-2, 2)$.

The test statistic T_N is independent of V under both regimes. Indeed, $\text{Cov}(X_1 + X_2, X_1 - X_2) = \text{Var}(X_1) - \text{Var}(X_2) = 0$, so $X_1 + X_2$ and $X_1 - X_2$ are independent; hence, $X_1 + X_2$ is independent of $|X_1 - X_2|$.

However, the mapping $D \mapsto (T_N, V)$ is not one to one: For fixed $t \in \mathbb{R}$ and $v > 0$, the constraints $X_1 + X_2 = t$ and $|X_1 - X_2| = v$ admit both cases $X_1 - X_2 = v$ and $X_1 - X_2 = -v$, which correspond to distinct values of (X_1, X_2) while producing the same $(T_N, V) = (t, v)$.

Finally, under both H_0 and H_1 one has $T_N = X_1 + X_2 \sim N(0, 2)$, so T_N cannot be MP. In contrast, the MP test statistic is the log-likelihood ratio $2(X_2 - X_1)$.

Example 3 (The third requirement in Proposition 2 fails although T_N is independent of V ; T_N is not MP) Let $D = (X, Y) \in \mathbb{R}^2$, and assume that X and Y are independent under both H_0 and H_1 . Assume that $Y \sim N(0, 1)$ under both regimes, and define the ancillary statistic $V = Y$. Let $T_N = X$. The mapping $D \mapsto (T_N, V) = (X, Y)$ is injective with measurable inverse.

Consider the simple hypotheses $H_0 : X \sim 0.5N(2, 1) + 0.5N(-2, 1)$ versus $H_1 : X \sim N(0, 1)$, and write $\phi(\cdot)$ for the $N(0, 1)$ density. Then, $f_0^X(x) = 0.5\phi(x - 2) + 0.5\phi(x + 2)$ and $f_1^X(x) = \phi(x)$, and we have the ratio

$$\frac{f_1^X(X)}{f_0^X(X)} = \frac{2\phi(X)}{\phi(X - 2) + \phi(X + 2)} = \frac{e^2}{\cosh(2X)}.$$

This ratio is not a monotone function of X . (It is even and is maximized at $X = 0$.) Therefore, the third requirement in Proposition 2 is not satisfied.

The Neyman–Pearson (hence MP) test rejects H_0 for large values of the likelihood ratio $\Lambda(D)$. The rule “reject H_0 for large values of $T_N = X$ ” is not MP in this example, even though T_N is independent of V and V is ancillary.

To state a result in the spirit of Basu’s theorem, we formulate the following proposition.

Proposition 3 Assume the test statistic T_N is complete under H_k , for $k = 0$ and/or $k = 1$. If T_N is MP, then T_N is independent of V under both H_0 and H_1 .

The completeness assumption above is used in the same way as in Basu-type arguments (see Basu’s theorem and related discussion in Ghosh (2002), and also Lehmann and Casella (2006)). Completeness ensures that a zero-mean function of T_N must be zero almost surely. Therefore, once one obtains an identity showing that the conditional law of V given T_N differs from the marginal law only through such a zero-mean function of T_N , completeness forces this difference to vanish. Equivalently, the conditional distribution of V given T_N coincides with the marginal distribution of V , and hence, V and T_N are independent. A complete proof with full details is provided in the Appendix. In Proposition 3, completeness is invoked precisely to justify this converse step.

Example 4 (*Tests of the location in symmetric models*) Assume we observe i.i.d. data points X_1, X_2, \dots, X_n that provide $D = \{X_1, \dots, X_n\}$. Let X_1 have a symmetric distribution with characteristic function $\phi_X(t) = 0$, for all $t \in \mathbb{R}^1$. Define $\bar{X}_n = \sum_{i=1}^n X_i/n$, and $\sigma^2 = \text{var}(X_1)$. We consider the problem of testing $H_0 : \mu = 0$ against $H_1 : \mu > 0$, where μ is the center of symmetry such that $f^{X_1}(\mu + u) = f^{X_1}(\mu - u)$ for every $u \in \mathbb{R}^1$, and where f^{X_1} denotes the unimodal density function of X_1 . A commonly used test statistic in this setting is $T = n^{0.5} \bar{X}_n/\sigma$.

According to Behboodian (1990) and Hamedani and Volkmer (2003), X_1, \dots, X_n are symmetric if and only if $X_j - \bar{X}_n$ is symmetric for some $j \in \{1, \dots, n\}$. Given $X_j - \bar{X}_n, j \in \{1, \dots, n\}$, and T , we can reconstruct D , since $X_j = X_j - \bar{X}_n + \sigma T/n^{0.5}, j \in \{1, \dots, n\}$.

Define a linear combination $Z_n = a_1(X_1 - \bar{X}_n) + \dots + a_n(X_n - \bar{X}_n)$, where $a_k \in \mathbb{R}^1, k \in \{1, \dots, n\}$. We have $Z_n = \sum_{i=1}^n X_i \left(a_i - \sum_{j=1}^n a_j/n \right)$, and thus, $\text{cov}(\bar{X}_n, Z_n) = E \sum_{i=1}^n (X_i - \mu)^2 \left(a_i - \sum_{j=1}^n a_j/n \right) = 0$.

By virtue of the Cramér–Wold theorem (i.e., since any linear combination of the components of the vector $\mathbf{V} = X_1 - \bar{X}_n, \dots, X_n - \bar{X}_n$ is uncorrelated with T), it follows that T and \mathbf{V} are independent when X_1 is normally distributed, where the operator T denotes transposition. The ratio $L(u) = f_1^T(u)/f_0^T(u)$ is exponential in u , due to the normality of T under both hypotheses. This satisfies the condition $f_1^{L(T)}(u) = u f_0^{L(T)}(u)$ required by Proposition 2. For a technical illustration of this principle, see Example 5. Thus, the conditions of Proposition 2 are satisfied.

We can also note that if X_1 is normally distributed and σ is known, then T is a complete statistic [Lehmann and Casella 2006, see Theorem 6.22].

Building on Example 4, we present a result that can characterize the normal distribution. Suppose the vector \mathbf{V} is ancillary with respect to the location parameter μ ; see, for example, Welsh (2011, p. 160) for discussion. This condition can also be treated in terms of the characteristic function of \mathbf{V} , denoted by

$$\begin{aligned} \phi_{\mathbf{V}}(t_1, \dots, t_n) &= E \exp \left\{ i \sum_{j=1}^n t_j (X_j - \bar{X}_n) \right\} \\ &= E \exp \left\{ i \sum_{j=1}^n t_j X_j - i \sum_{j=1}^n t_j \sum_{k=1}^n X_k/n \right\} \\ &= E \exp \left\{ i \sum_{j=1}^n X_j \left(t_j - \sum_{k=1}^n t_k/n \right) \right\} = \prod_{j=1}^n \phi_X \left(t_j - \sum_{k=1}^n t_k/n \right), \end{aligned}$$

where $i^2 = -1, t_k \in \mathbb{R}^1$ for $k \in \{1, \dots, n\}$, and ϕ_X , the characteristic function of X_1 , is assumed to satisfy $\phi_X(\mu + t) = \phi_X(\mu - t)$ for all $t \in \mathbb{R}^1$.

Proposition 4 *Let X_1, X_2, \dots, X_n be i.i.d. random variables such that X_1 has a symmetric unimodal distribution about μ and satisfies $E X_1^2 < \infty$. Suppose that the sample mean \bar{X}_n is a complete statistic under either H_0 or H_1 . Then, X_1 follows a normal*

distribution if and only if $T = n^{0.5}\bar{X}_n/\sigma$ is the MP test statistic for the location parameter.

It is worth noting that, in a broad class of models, statistics based on \bar{X}_n may serve as MP test statistics. For instance, consider testing the null hypothesis H_0 that X_1, \dots, X_n are i.i.d. observations from a density function f_0 , against the alternative $H_1 : X_1, \dots, X_n$ are i.i.d. from $f_1(u) = \exp(\theta u + \gamma) f_0(u)$, where θ and γ are unknown; see Vexler and Hutson (2018, pp. 97–98).

An additional characterization of the distribution via the form of an MP test is presented in Example 5 [see also Pfanzagl (1968) in this context].

For testing the hypotheses considered in Example 4, the statistics commonly used in practice are $T = n^{0.5}\bar{X}_n/\sigma$ and the Wilcoxon signed rank test statistic. Proposition 4 raises a concern regarding the comparative performance of these procedures: The t -test may be superior to the Wilcoxon test when the underlying distribution deviates from normality; see Vexler and Yu (2018) for discussion in this context. Rosenblatt and Benjamini (2013) showed that the Wilcoxon signed rank test is more efficient than the T -based test under the alternative hypothesis $H_1 : 0 < \mu \leq 1$, where $X_1 \sim (1 - \mu)N(0, 1) + \mu N(\theta, \sigma^2)$ and the parameters θ and σ are known. Furthermore, Shiraishi (1986) proved that the asymptotic power of the Wilcoxon signed rank test equals that of the MP test under the local alternative $H_1 : X_1 \sim (1 - \mu)G(x) + \mu\{G(x)\}^2$, where $\mu = \delta/n^{0.5}$, $G(x)$ is a known logistic distribution, and δ is fixed. Shiraishi (1986) also concluded that the numerical values of the asymptotic relative efficiency of the Wilcoxon test with respect to the T -based test show no loss, even under general contaminated alternatives.

Example 5 (*Tests of the location in an asymmetric exponential model*) Let X_1, X_2, \dots, X_n be a sample from the negative exponential distribution $F_e(x) = 1 - \exp\{-x/\sigma\}$ ($x > 0$), with location parameter μ and scale parameter $\sigma > 0$. Consider testing hypotheses concerning μ . Let $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ denote the order statistics based on $\{X_1, \dots, X_n\}$. Define the vector $\mathbf{V} = [V_1, \dots, V_{n-1}]$, where $V_i = X_{(i+1)} - X_{(i)}$ for $i \in [1, \dots, n - 1]$. The vector \mathbf{V} is ancillary (Buehler 1982), since

$$\begin{aligned} \Pr \left\{ \bigcap_{i=1}^{n-1} \{V_i < v_i\} \right\} &= \int_{x_1 < \dots < x_n} I \left\{ \bigcap_{i=1}^{n-1} \{x_{i+1} - x_i < v_i\} \right\} n! \prod_{i=1}^n f^{X_1}(x_i) dx_1 \dots dx_n \\ &= \int_{z_1 < \dots < z_n} I \left\{ \bigcap_{i=1}^{n-1} \{z_{i+1} - z_i < v_i\} \right\} n! \prod_{i=1}^n f^{X_1}(z_i + \mu) dz_1 \dots dz_n, \end{aligned}$$

where f^{X_1} is the density function of X_1 . Given $X_{(1)}$ and \mathbf{V} , the original sample can be reconstructed via the recursion: $X_{(2)} = V_1 + X_{(1)}$, $X_{(3)} = V_2 + X_{(2)}$, \dots , $X_{(n)} = V_{n-1} + X_{(n-1)}$. (Ferguson 1967) established the independence between the statistic $T = X_{(1)}$ and the vector \mathbf{V} . Assume, for simplicity, that when testing the location parameter μ , we consider the null hypothesis $H_0 : \mu = \mu_0$, and let $\mu_1 > \mu_0$ be a possible alternative value. The ratio

$$L(u) = f_1^T(u)/f_0^T(u) = e^{(\mu_0 - \mu_1)u/\sigma} h(u)^{n-1}, \quad \text{for } u > \mu_1 > \mu_0,$$

with

$$h(u) = \frac{1 - e^{-(u-\mu_1)/\sigma}}{1 - e^{-(u-\mu_0)/\sigma}}, \quad h(u) = \frac{e^{-(u-\mu_1)/\sigma} - e^{-(u-\mu_0)/\sigma}}{(1 - e^{-(u-\mu_0)/\sigma})^2} > 0.$$

Thus, $L(u)$ is strictly increasing. In a manner similar to Remark 3, we may consider the monotonic transformation $T_N = L(T)$ as a theoretical device, which satisfies $f_1^{T_N}(u) = u f_0^{T_N}(u)$.

Then, we conclude that the conditions of Proposition 2 hold, and therefore, T is an MP test statistic.

Now, suppose X_1, \dots, X_n are i.i.d. observations from a continuous distribution. Redefine the ancillary vector $\mathbf{V} = [X_{(2)} - X_{(1)}, X_{(3)} - X_{(1)}, \dots, X_{(n)} - X_{(1)}]$. According to Govindarajulu (1966), T and \mathbf{V} are independent if and only if X_1 follows the negative exponential distribution $F_e(x)$. Thus, if $T = X_{(1)}$ is an MP test statistic for the location, then X_1, \dots, X_n must follow the negative exponential distribution. This yields a characterization of the negative exponential distribution via the form of the MP test statistic in this decision-making problem.

Example 6 (Tests of the scale (rate) of a gamma distribution) Let observations X_1, \dots, X_n be i.i.d. from a gamma distribution with the scale parameter θ (the rate parameter equals $1/\theta$) and the shape parameter κ . The random variables $X_i/\bar{X}_n = (\theta Y_i)/(\theta Z) = Y_i/Z$, where $Y_i \sim \text{Gamma}(\kappa, 1)$ and $Z \sim \text{Gamma}(n\kappa, 1/n)$, $i \in [1, \dots, n]$. The distribution of the vector $\mathbf{V} = [X_1/\bar{X}_n, \dots, X_n/\bar{X}_n]$ is Dirichlet with parameter vector $[\kappa, \dots, \kappa]$, and it does not depend on θ .

According to Lin and Stoyanov (2022), the statistics \bar{X}_n and \mathbf{V} are independent. When $T = \bar{X}_n \sim \text{Gamma}(n\kappa, \theta/n)$, the analysis of the ratio $L(u) = f_1^T(u)/f_0^T(u)$ is the same as in Example 2.1. Therefore, in this setting, \bar{X}_n serves as an MP test statistic for the scale parameter θ .

In relation to Proposition 2, a discussion on using distance correlation to compare test statistics with respect to their dependence on ancillary vectors is provided in the Supplementary Material (see Remark S1.2).

4 Distribution-free applications

This section presents simple examples that demonstrate the proposed technique in a transparent and practically applicable manner. The following analysis examines ways to adapt existing test mechanisms and to compare alternative decision-making procedures using the framework developed in Sect. 3.

In this section, we use ancillarity in an approximate sense (see Remark 2 for this interpretation). Accordingly, the ancillary invariance should be understood as approximate in finite samples, while still serving as a useful design principle for constructing and comparing the testing procedures.

Section 4.1 explores a simple modification of classical tests for normality. We show that the modified tests are asymptotically twice as efficient as their original counterparts under symmetric alternatives. The example is clear and self-contained, and it highlights elements that may extend to broader contexts of goodness-of-fit testing.

Section 4.2 addresses a multivariate mean testing problem, comparing Hotelling's procedure with a simple trace-normalized test within the ancillary-independence framework. The results show that the trace-normalized test can outperform Hotelling's procedure under heavy-tailed distributions, while Hotelling's test remains preferable under normality.

4.1 A straightforward approach to improving goodness-of-fit tests

We consider the problem of testing for normality based on a sample X_1, \dots, X_n . There is a body of literature on tests for normality against asymmetric alternatives (Lin and Mudholkar 1980) and on tests for normality against symmetric alternatives (Spiegelhalter 1977). We illustrate an application of the presented framework to tests for symmetric alternatives. Let the underlying distribution of X_i , $i \in [1, \dots, n]$, be denoted by $F(u)$, which is assumed to be unknown.

Ancillary representation According to Proposition 2, the vector $\mathbf{V}_1 = [F(|X_1|) - F(-|X_1|), \dots, F(|X_n|) - F(-|X_n|)]$ serves as an example of ancillary statistics that can be particularly effective for constructing a test of normality, assuming that $F(u)$ were known, since $\Pr(|X_1| \leq u) = F(u) - F(-u)$. The vectors \mathbf{V}_1 and $[\text{sign}(X_1), \dots, \text{sign}(X_n)]$ together can be used to reconstruct the original data (X_1, \dots, X_n) . Using the empirical estimator of $\Pr(|X_1| \leq u)$ based on $|X_1|, \dots, |X_n|$, we approximate \mathbf{V}_1 by $\hat{\mathbf{V}} = n^{-1} [R_1^+, \dots, R_n^+]$, where R_i^+ denotes the rank of $|X_i|$, $i \in [1, \dots, n]$.

Define the order statistics $|X|_{(1)}, \dots, |X|_{(n)}$ based on $|X_1|, \dots, |X_n|$. We invoke the following result: For any distribution F symmetric about zero, the random vectors $\hat{\mathbf{V}}$, $[\text{sign}(X_1), \dots, \text{sign}(X_n)]$, and $[|X|_{(1)}, \dots, |X|_{(n)}]$ are mutually independent [Hájek et al. 1967, p. 40].

This independence, along with Proposition 2, motivates the development of tests for normality against symmetric alternatives based on the observations $|X|_{(1)}, \dots, |X|_{(n)}$.

Modified procedures For example, the Anderson–Darling test (AD), the Cramér–von Mises test (CvM), and the Kolmogorov–Smirnov test (KS) involve the empirical distribution function of X_1, \dots, X_n . We propose modifications to these tests based on the following transformation: Define $Z_i = (X_i - \bar{X}_n) \left\{ n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2 \right\}^{-0.5}$, with $\bar{X}_n = \sum_{i=1}^n X_i/n$, and apply the AD, CvM, and KS test strategies based on $|Z|_{(1)}, \dots, |Z|_{(n)}$ for assessing the null hypothesis $\Pr(|Z| < u) = 2\Phi(u) - 1$, where $\Phi(u) = (2\pi)^{-0.5} \int_{-\infty}^u \exp(-t^2/2) dt$, resulting in the modified versions of the tests, denoted as mAD, mCvM, and mKS, respectively. Note that it is common for the classical AD, CvM, and KS tests for normality to be based on Z_1, \dots, Z_n , in order to control their type I error rates when the expectation and variance of the observations X_1, \dots, X_n are unknown. Regarding the equivalence between the statements $Z \sim N(0, 1)$ and $\Pr\{|Z| \leq u\} = 2\Phi(u) - 1$ for all $u \geq 0$ under the assumption that Z is symmetrically distributed, we refer the reader to Vexler (2020).

Asymptotic relative efficiency The ancillarity-based framework considered in Sect. 3 guides us to employ absolute values of the observations when applying the AD and

CvM tests for normality. This, in turn, motivates an analysis of their asymptotic efficiency in the sense of Inglot et al. (2000). It can be shown that the modified Anderson–Darling and modified Cramér–von Mises tests are asymptotically twice as efficient as their original counterparts under symmetric alternatives. The corresponding proof is given in Section S2 of the Supplementary Material.

Numerical illustration The suggested approach above uses an empirical approximation of the theoretical principle shown in Proposition 2. Its efficiency should be evaluated thoroughly, for example, through extensive Monte Carlo simulations. In this regard, we note that our attempts to modify the Shapiro–Wilk test, for example, by incorporating the expectations of $|Z|_{(i)}$, $i \geq 1$, or employing L-estimators based on $|Z|_{(1)}, \dots, |Z|_{(n)}$ in the SW manner, did not result in noticeable improvements. Motivated by empirical evaluations, the modification (mSW) of the SW test that we suggest is as follows: Consider the data $D = \{-|Z|_{(1)}, \dots, -|Z|_{(n)}, |Z|_{(1)}, \dots, |Z|_{(n)}\}$, and compute the classic SW test statistic based on D . Under H_0 , the mSW statistic is distributed independently of the parameters of the null distribution of the observations.

To evaluate the extent of power loss when the proposed modifications are applied to asymmetric alternatives, we include the Spiegelhalter test (S) for comparison, as it is specifically constructed for testing normality against symmetric alternatives (Spiegelhalter 1977).

Under H_0 , the considered test statistics have null distributions that are free of the unknown parameters. Hence, we obtain their critical values by Monte Carlo simulation under $X_i \sim N(0, 1)$, $i \geq 1$. Specifically, for each value of n , the critical values for the SW, mSW, AD, mAD, CvM, mCvM, and S statistics were computed from their empirical sampling distributions based on 50,000 standard normal samples.

Table 1 summarizes selected Monte Carlo power results for various alternatives at the 5% significance level, based on 50,000 replications per setting. Sample sizes are indicated in the alternative labels when different from $n = 30$. The table highlights that the modified tests tend to outperform their classical counterparts under symmetric alternatives, while remaining relatively competitive under asymmetric settings. We note, however, that under strongly skewed alternatives the ancillary-guided construction may reduce sensitivity to the very features driving the departure from normality. This loss of power is of the same order as that of the classical test S and is most visible for highly asymmetric laws (e.g., $\text{Exp}(1)$). In such regimes, the guiding ancillary structure is far from being approximately valid. For example, in the case where the observations follow a Beta(2, 2) distribution, the relative improvements in power were: 160.5%, 31.2%, 79.7%, and 62.2%, corresponding to the SW, AD, CvM, and KS test mechanisms, respectively. In several cases, we observed that the modification led to situations in which the original counterparts were biased, whereas their transformed versions consistently achieved power exceeding 5%. For instance, consider the scenario where $X = y_1 - y_2$, with $y_k \sim \text{Beta}(0.5, 0.5)$.

In a few cases, the classical S test outperforms the modified tests. However, there are scenarios where the S test seems to be inconsistent. For example, Table 1 demonstrates such situations when $n = 50, 100, 150$ are considered, and the observations come from the mixture of normal distributions $N(-1, 1)$ and $N(1, 1)$. In these cases, the proposed tests display robust power characteristics.

Table 1 Monte Carlo power of normality tests under various alternatives at $\alpha = 0.05$

Alternative	SW	mSW	AD	mAD	CvM	mCvM	KS	mKS	S
$X \sim N(1, 3^2)$	0.051	0.047	0.049	0.050	0.048	0.050	0.048	0.048	0.048
$X \sim \text{Exp}(1)$	0.968	0.514	0.936	0.334	0.896	0.348	0.837	0.394	0.341
$X = y_1 - y_2, y_k \sim \text{Exp}(1)$	0.358	0.358	0.371	0.381	0.356	0.386	0.268	0.349	0.437
$X \sim \text{Beta}(0.5, 0.5)$	0.943	0.944	0.858	0.856	0.741	0.816	0.475	0.723	0.997
$X = y_1 - y_2, y_k \sim \text{Beta}(0.5, 0.5)$	0.056	0.182	0.052	0.083	0.044	0.078	0.047	0.078	0.156
$X \sim \text{Beta}(2, 5)$	0.275	0.09	0.225	0.067	0.204	0.075	0.242	0.066	0.088
$X = y_1 - y_2, y_k \sim \text{Beta}(2, 5)$	0.043	0.043	0.047	0.049	0.048	0.047	0.053	0.055	0.048
$X \sim \text{Beta}(2, 2)$	0.076	0.198	0.076	0.100	0.069	0.124	0.061	0.099	0.145
$X = y_1 - y_2, y_k \sim \text{Beta}(2, 2), n = 30$	0.036	0.061	0.043	0.048	0.042	0.051	0.045	0.049	0.044
$X = y_1 - y_2, y_k \sim \text{Beta}(2, 2), n = 50$	0.038	0.073	0.047	0.055	0.048	0.059	0.047	0.054	0.036
$X = y_1 - y_2, y_k \sim \text{Beta}(2, 2), n = 150$	0.071	0.179	0.07	0.09	0.067	0.096	0.06	0.085	0.005
$X \sim \text{Gamma}(3, 1)$	0.562	0.202	0.478	0.106	0.434	0.107	0.433	0.093	0.147
$X = y_1 - y_2, y_k \sim \text{Gamma}(3, 1)$	0.131	0.141	0.118	0.115	0.107	0.121	0.085	0.111	0.138
$X \sim \chi_1^2$	0.999	0.828	0.999	0.741	0.996	0.777	0.985	0.851	0.558
$X = y_1 - y_2, y_k \sim \chi_1^2$	0.665	0.699	0.729	0.753	0.736	0.737	0.617	0.741	0.766
$X \sim \text{Triangular}(-1, 1), n = 30$	0.035	0.079	0.041	0.051	0.039	0.052	0.042	0.049	0.063
$X \sim \text{Triangular}(-1, 1), n = 150$	0.214	0.538	0.109	0.134	0.077	0.132	0.057	0.107	0.021
$X \sim 0.5N(-1, 1) + 0.5N(1, 1), n = 30$	0.051	0.098	0.057	0.079	0.062	0.086	0.057	0.077	0.063
$X \sim 0.5N(-1, 1) + 0.5N(1, 1), n = 100$	0.110	0.227	0.131	0.241	0.134	0.255	0.101	0.178	0.016
$X = U[-1, 1] + N(0, 0.5^2), n = 30$	0.042	0.074	0.048	0.054	0.051	0.059	0.051	0.059	0.050
$X = U[-1, 1] + N(0, 0.5^2), n = 50$	0.045	0.084	0.052	0.064	0.054	0.072	0.051	0.068	0.038
$X = U[-1, 1] + N(0, 0.5^2), n = 150$	0.087	0.184	0.100	0.137	0.092	0.151	0.077	0.13	0.003

The mSW, mAD, mCvM, and mKS denote modified tests. Under symmetric alternatives, the modified tests are expected to outperform their classical counterparts. Under asymmetric settings, any potential loss in power of the modified tests may be compared to that of the classical S test, which is designed for testing normality against symmetric alternatives

Table 1 indicates that the proposed ancillary-guided modifications primarily act as symmetry-focused power concentrators. For symmetric alternatives, the modified tests often deliver substantial power gains over their classical counterparts, reflecting reduced ancillary leakage and increased sensitivity to the departure features targeted by the construction.

For every test considered, the nominal $\alpha = 0.05$ lies within the 95% confidence interval $\hat{\alpha} \pm 1.96 \{ \hat{\alpha}(1 - \hat{\alpha})/50,000 \}^{0.5}$, confirming that the tests maintain the intended significance level within expected sampling variation.

To provide a practical interpretation, we also report an “ n -equivalent” comparison for representative symmetric alternatives. For a fixed target power level, we determine the smallest sample size needed by the classical test to match the power of the modified test at the reported n , thereby quantifying the effective sample-size savings implied in Table 1. For instance, under the symmetric Beta(2, 2) alternative with $n = 30$, mSW achieves approximately 19.8% power, whereas SW reaches 7.6%. To attain 19.8% power, SW would require roughly $n \approx 80$ observations, corresponding to a reduction of about 50 observations (approximately 63% fewer) when using mSW. Under the alternative $X \sim 0.5N(-1, 1) + 0.5N(1, 1)$, at $n = 100$ Table 1 reports experimental powers 0.131 for AD and 0.241 for mAD. Under the Monte Carlo design described in this section, we find that AD attains empirical power 0.241 at about $n = 180$. Thus, in this setting mAD at $n = 100$ is roughly comparable to AD at about $n \approx 180$, corresponding to a sample-size reduction of about 44%.

Overall, the proposed modifications are simple to implement, and the experimental results confirm that the resulting procedures are effective in detecting deviations from normality under symmetry while maintaining robustness under broader alternatives.

Remark 6 Our method requires identifying relevant ancillary statistics. In this context, we note that the concept of ancillarity can be treated through appropriate approximations (Skovgaard 1985). In many situations, maximum log-likelihood ratio-type statistics may follow a χ^2 asymptotic distribution independently of the underlying data distribution. In nonparametric settings, the Wilcoxon test statistic based on the differences $Y_1 = X_1 - X_2, Y_2 = X_3 - X_4, \dots$ can serve as an example of an ancillary statistic.

4.2 Multivariate example: Hotelling and Trace-Normalized test statistics

Test statistics Let $\mathbf{X}_1, \dots, \mathbf{X}_n$ be i.i.d. p -dimensional random vectors with mean $\mu = E(\mathbf{X}_i)$ and covariance matrix $\Sigma = \text{Cov}(\mathbf{X}_i)$. We consider the one-sample problem: $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$. In this context, the classical Hotelling test statistic is

$$T^2 = n \bar{\mathbf{X}} S^{-1} \bar{\mathbf{X}}, \quad \bar{\mathbf{X}} = n^{-1} \sum_{i=1}^n \mathbf{X}_i, \quad S = (n - 1)^{-1} \sum_{i=1}^n (\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})'$$

An alternative test statistic, useful particularly when p is relatively large, is the trace-normalized form

$$T_1^2 = \frac{n p \bar{\mathbf{X}} \bar{\mathbf{X}}}{\text{tr}(S)}.$$

The Hotelling T^2 test is routinely used under multivariate normality of the observed data and also in broader non-normal settings. When inversion of S is unstable or computationally burdensome, the T_1^2 -based procedure provides a practical relaxation by replacing S^{-1} with the scalar adjustment $\text{tr}(S)^{-1}$; see Fujikoshi et al. (2004).

Ancillary structure In a nonparametric setting, and within the guidance of the theoretical framework of this paper, we ask under what conditions the use of T^2 or T_1^2 can be recommended for fixed, but possibly large, p . To this end, we define the residual matrix $V = [\mathbf{X}_1 - \bar{\mathbf{X}}, \dots, \mathbf{X}_n - \bar{\mathbf{X}}] \in \mathbb{R}^{n \times p}$, the Gram matrix $W = VV^T$, and note $S = (n-1)^{-1}V^T V$. Let $H = I_n - n^{-1}\mathbf{1}\mathbf{1}^T$ denote the centering matrix, so that $V = HX$ and $W = HXX^T H$, where X is the $n \times p$ data matrix with rows \mathbf{X}_i . In this framework, the matrix V of centered observations is assumed to be an ancillary statistic; that is, its distribution does not depend on the mean parameter μ under either hypothesis.

As an illustrative model, suppose that $\mathbf{X}_i = \mu + \Gamma \mathbf{Z}_i$, $i = 1, \dots, n$, where Γ is a fixed $p \times p$ matrix and the \mathbf{Z}_i are i.i.d. random vectors with $E(\mathbf{Z}_i) = 0$. It is assumed that Γ and the distribution of \mathbf{Z}_1 do not depend on μ .

Dependence measure To assess and compare the dependence of the statistics T^2 and T_1^2 on the ancillary structure, we employ the vector correlation (RV) coefficient introduced by Escoufier (1973). For a scalar U and a random vector \mathbf{Y} ,

$$\text{RV}(U, \mathbf{Y}) = \frac{\|\text{Cov}(U, \mathbf{Y})\|^2}{\text{Var}(U) \|\text{Var}(\mathbf{Y})\|} \in [0, 1],$$

where $\|\cdot\|$ denotes the Euclidean norm for vectors and the Frobenius norm for matrices. Then, $\text{RV}(U, \mathbf{Y}) = 0$ when U and \mathbf{Y} are independent, and $\text{RV}(U, \mathbf{Y}) = 1$ when they are perfectly linearly dependent. In what follows, we will use the RV coefficient to approximate and compare the dependence of T^2 and T_1^2 on the ancillary statistic V .

By Lemma S3.1 in the Supplementary Material (Sect. S3), since T^2 and T_1^2 are symmetric in the sample indices, we obtain

$$\text{RV}(T^2, V) = 0 \quad \text{and} \quad \text{RV}(T_1^2, V) = 0.$$

Consequently, we proceed to the next (quadratic) order by analyzing

$$\text{RV}(T^2, \text{vec}(W)) \quad \text{and} \quad \text{RV}(T_1^2, \text{vec}(W)),$$

in order to capture the leading nonzero quadratic component of the dependence between the test statistics and the ancillary structure encoded by V . Here, for any matrix $A \in \mathbb{R}^{m \times r}$, $\text{vec}(A) \in \mathbb{R}^{mr}$ denotes the column-stacked vectorization of A . In particular, since $V \in \mathbb{R}^{n \times p}$, we have $W = VV^T \in \mathbb{R}^{n \times n}$ and $\text{vec}(W) \in \mathbb{R}^{n^2}$.

To study the ratio

$$R = \frac{RV(T_1^2, \text{vec}(W))}{RV(T^2, \text{vec}(W))},$$

we define the sphericity index $\psi = p \text{tr}(\Sigma^2) \{ \text{tr}(\Sigma) \}^{-2}$. By the Cauchy–Schwarz inequality, we have $\psi \geq 1$ with equality iff $\Sigma \propto I_p$. We also denote an elliptical kurtosis factor as follows. Assume the elliptical fourth-moment identity

$$E \left\{ (Y - BY)(Y - CY) \right\} = 2 \text{tr}(B \Sigma C \Sigma) + (\theta - 1) \text{tr}(B \Sigma) \text{tr}(C \Sigma), \quad Y = \mathbf{X}_1 - \mu,$$

holds for symmetric $p \times p$ matrices B, C . The factor $\theta \geq 1$ is called the *elliptical kurtosis factor*. It links directly to Mardia’s multivariate kurtosis $\beta_{2,p} = E \{ (X - \mu) \Sigma^{-1} (X - \mu) \}^2$, via $\theta = \beta_{2,p} / \{ p(p + 2) \}$ under the illustrative model mentioned above, e.g., Mardia (1974). For example, when $\mathbf{X}_1 \sim \mathcal{N}_p(\mu, \Sigma)$, $\beta_{2,p} = p(p + 2)$ so $\theta = 1$; when $\mathbf{X}_1 \sim$ multivariate $t_\nu(\mu, \Sigma)$ with $\nu > 4$, we have $\theta = (\nu - 2) / (\nu - 4) > 1$.

Asymptotic behavior We obtain the asymptotic approximation of R in the form

$$R = \frac{(2\psi + (\theta - 1)p)^2}{\psi (2 + (\theta - 1)p)^2} + o(1), \quad n \rightarrow \infty,$$

where the following regimes can be considered: (i) when p is fixed, the remainder in the ratio expansion is of order $O(n^{-1})$; (ii) when both $p, n \rightarrow \infty$ proportionally with $p/n \rightarrow c \in (0, 1)$, the remainder is larger, of order $O(n^{-1/2})$.

A corresponding full derivation is given in Section S3 of the Supplementary Material.

Under Gaussianity, $R \approx \psi \geq 1$, reflecting the well-known optimality of the T^2 decision-making procedure in this case. With heavy tails ($\theta > 1$), $(\theta - 1)p$ dominates and R can fall below 1. This indicates that the trace-normalized T_1^2 may reduce ancillary dependence relative to T^2 , and in such strongly heavy-tailed settings T_1^2 can be superior to T^2 .

Numerical illustration To provide a brief numerical study, we considered $n = 25$ and $p = 15$. In the first design, data were generated from $N_p(\mu, \Sigma)$ with $\mu = (0, 0.9, 0, \dots, 0)$ under H_1 and Σ given by $\Sigma_{ii} = 1$ for all i and $\Sigma_{11} = 10$. In this case, the dependence ratio was estimated as $\widehat{R} \approx 1.4 > 1$. A Monte Carlo experiment with 30,000 replications and empirical critical values at the 5% level yielded power estimates of about 0.41 for Hotelling’s T^2 and 0.22 for the trace-normalized statistic T_1^2 .

As a contrasting heavy-tailed design, we drew data from a multivariate t_5 distribution with mean vector $\mu_t = (0.9, 0, \dots, 0)$ under H_1 and covariance matrix equal to the identity. Here, the estimated ratio was $\widehat{R} \approx 0.7 < 1$, and the same simulation scheme as in the normal case produced powers of about 0.57 for T^2 and 0.84 for T_1^2 .

These power differences have direct practical implications. Under normality, Hotelling’s T^2 is more powerful, confirming its optimality in the regime for which

it was designed. By contrast, under heavy tails the trace-normalized T_1^2 can be substantially stronger at the same sample size. The observed gain—from 0.57 to 0.84—corresponds to an absolute increase of about 0.27 in the probability of detecting the mean shift. In this heavy-tailed regime, Hotelling's T^2 would require roughly 1.5–2 times as many observations (with $n \in (40, \dots, 50)$) to match the power achieved by T_1^2 at $n = 25$.

This brief numerical study agrees with the theoretical prediction: T^2 is preferable in the $R > 1$ regime, whereas T_1^2 is advantageous when $R < 1$.

5 Conclusion

This paper has introduced a relativity-based perspective for constructing, comparing, and improving test statistics by exploiting approximate independence from ancillary structures. Within a decision-theoretic framework, we have shown that reducing the dependence between a test statistic and a vector of ancillary statistics can yield MP procedures under suitable conditions. Both direct and converse results in the spirit of Basu's theorem were established, offering a structural characterization of MP test statistics through their independence from ancillary quantities. We also demonstrated that certain forms of MP test statistics implicitly characterize the data distribution.

The practical utility of the proposed approach was illustrated through simple examples and two nonparametric applications. We developed ancillary-guided modifications of classical normality tests, demonstrating their superiority in terms of asymptotic relative efficiency and supported by simulation studies, which show consistent power improvements under symmetric alternatives. We examined a multivariate mean testing problem, comparing Hotelling's and trace-normalized statistics, and clarified the conditions under which each procedure is preferable. These procedures are straightforward to implement and highlight the effectiveness of the ancillary-independence framework in practice.

Several directions merit further investigation. Extending the methodology to multiple testing, high-dimensional inference, regression, and model selection settings may reveal broader connections between ancillarity and the efficiency of statistical decision-making procedures. Within the proposed framework, alternative dependence measures—such as distance correlation, mutual information, or maximal correlation—may provide new perspectives on test construction. Data-driven approaches for selecting ancillary-guided transformations—potentially involving resampling or empirical likelihood—offer promising opportunities for future work.

Appendix

Proof of Proposition 1

Proof Consider the elementary inequality: For all real values B and C ,

$$(B - C) \{I(B \geq C) - \delta\} \geq 0,$$

where $I(\cdot)$ is the indicator function and $\delta \in [0, 1]$. Let $\delta = I(T \geq C^T)$ denote the rejection rule for a test based on the statistic T at threshold C^T ; that is, we reject H_0 when $\delta = 1$. Taking $B = L(T_N)$, we obtain:

$$E_0 \{(L(T_N) - C)I(L(T_N) \geq C)\} \geq E_0 \{(L(T_N) - C)I(T \geq C^T)\}.$$

Now, fix thresholds C and C^T such that the type I error rates are equal: $\Pr_0\{L(T_N) \geq C\} = \Pr_0\{T \geq C^T\}$. It follows that:

$$E_0 [I(L(T_N) \geq C)L(T_N)] \geq E_0 [I(T \geq C^T)L(T_N)].$$

Since the densities satisfy the identity $uf_0^{L(T_N)}(u) = f_1^{L(T_N)}(u)$, the left-hand side becomes:

$$\begin{aligned} E_0 [I(L(T_N) \geq C)L(T_N)] &= \int I(u \geq C)uf_0^{L(T_N)}(u) du = \int I(u \geq C)f_1^{L(T_N)}(u) du \\ &= \Pr_1 \{L(T_N) \geq C\}. \end{aligned}$$

To compute the right-hand side, recall that $T = \psi(T_N, A)$, and use the independence of T_N and A under H_0 :

$$\begin{aligned} E_0 [I(T \geq C^T)L(T_N)] &= E_0 [I(\psi(T_N, A) \geq C^T)L(T_N)] \\ &= \iint I(\psi(W(u), a) \geq C^T)uf_0^{L(T_N),A}(u, a) du da \\ &= \iint I(\psi(W(u), a) \geq C^T)uf_0^{L(T_N)}(u) f_0^A(a) du da \\ &= \iint I(\psi(W(u), a) \geq C^T)f_1^{L(T_N)}(u) f_0^A(a) du da \end{aligned}$$

where W denotes the inverse of L . Then, applying the fact that $f_0^A = f_1^A$, we have

$$\begin{aligned} E_1 [I(T \geq C^T)L(T_N)] &= \iint I(\psi(W(u), a) \geq C^T)f_1^{L(T_N)}(u) f_1^A(a) du da \\ &= E_1 \int I(\psi(W(u), A) \geq C^T)f_1^{L(T_N)}(u) du \\ &= E_1 E_1 I(\psi(W(L(T_N)), A) \geq C^T) = E_1 E_1 I(T \geq C^T) \\ &= \Pr_1(T \geq C^T), \end{aligned}$$

This completes the proof. □

Proof related to Remark 3

In this setting, we follow the proof scheme of Proposition 1 and note that

$$\begin{aligned} E_0 [I(L(T_N) \geq C)L(T_N)] &= E_0 \left[I(L(T_N) \geq C) \frac{f_1^{L(T_N)}(L(T_N))}{f_0^{L(T_N)}(L(T_N))} \right] \\ &= \int I(u \geq C) \frac{f_1^{L(T_N)}(u)}{f_0^{L(T_N)}(u)} f_0^{L(T_N)}(u) du \\ &= \Pr_1 \{L(T_N) \geq C\} . \end{aligned}$$

Proof of Proposition 2

Proof Since $D \mapsto (T_N, V)$ is injective with a measurable inverse, and L is strictly monotonic, it follows that the mapping $D \mapsto (L(T_N), V)$ is injective with a measurable inverse. Thus, the MP test statistic can be written as $\Lambda(D) = g(L(T_N), V)$, for some bounded real-valued function g . (For example, $\Lambda(D)$ can be the corresponding likelihood ratio.)

Without loss of generality, let $L(u)$ be strictly increasing, and $Y = L(T_N)$. In a manner similar to the proof of Proposition 1, we have

$$E_0 [I(Y \geq C) Y] \geq E_0 [I(\Lambda \geq C^\Lambda) Y] ,$$

where C and C^Λ are chosen such that $\Pr_0(Y \geq C) = \Pr_0(\Lambda \geq C^\Lambda)$, i.e., the type I error rates $\Pr_0(T_N \geq C^{T_N}) = \Pr_0(\Lambda \geq C^\Lambda)$, where the test threshold $C^{T_N} = W(C)$, and W denotes the inverse of L .

Since $f_1^Y(u) = u f_0^Y(u)$, Y and V are independent, and $f_0^V = f_1^V$, it follows that

$$\begin{aligned} E_0 [I(Y \geq C)Y] &= \int I(u \geq C) u f_0^Y(u) du = \int I(u \geq C) f_1^Y(u) du \\ &= \Pr_1(Y \geq C) = \Pr_1(T_N \geq C^{T_N}), \end{aligned}$$

and

$$\begin{aligned} E_0 [I(\Lambda \geq C^\Lambda)Y] &= E_0 [I(g(Y, V) \geq C^\Lambda)Y] \\ &= \iint I(g(u, v) \geq C^\Lambda) u f_0^{Y,V}(u, v) du dv = \iint I(g(u, v) \geq C^\Lambda) u f_0^Y(u) f_0^V(v) du dv \\ &= \iint I(g(u, v) \geq C^\Lambda) f_1^Y(u) f_0^V(v) du dv = \iint I(g(u, v) \geq C^\Lambda) f_1^Y(u) f_1^V(v) du dv \\ &= E_1 \int I(g(u, V) \geq C^\Lambda) f_1^Y(u) du = E_1 E_1 I(g(Y, V) \geq C^\Lambda) = E_1 E_1 I(\Lambda \geq C^\Lambda) \\ &= \Pr_1(\Lambda \geq C^\Lambda). \end{aligned}$$

This completes the proof. □

Proof of Remark 4

Proof The proof is based on that of Proposition 2.

We begin proving that $f_1^{L(T_N)}(u) = u f_0^{L(T_N)}(u)$, for $u > 0$, when the conditions of Remark 4 are satisfied. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be any bounded measurable function. By the definition of the density $f_k^{L(T_N)}$, we have, for $k = 0, 1$,

$$\int_{\mathbb{R}} h(u) f_k^{L(T_N)}(u) du = E_k h(L(T_N(M))) = \int_{\mathbb{R}^p} h(L(T_N(m))) f_k^M(m) d\tau(m).$$

Then, for $k = 1$, we obtain

$$\begin{aligned} \int_{\mathbb{R}} h(u) f_1^{L(T_N)}(u) du &= \int_{\mathbb{R}^p} h(L(T_N(m))) f_1^M(m) d\tau(m) \\ &= \int_{\mathbb{R}^p} h(L(T_N(m))) L(T_N(m)) f_0^M(m) d\tau(m). \end{aligned}$$

On the other hand, under H_0 ,

$$\begin{aligned} \int_{\mathbb{R}} h(u) u f_0^{L(T_N)}(u) du &= E_0 L(T_N(M)) h(L(T_N(M))) \\ &= \int_{\mathbb{R}^p} L(T_N(m)) h(L(T_N(m))) f_0^M(m) d\tau(m). \end{aligned}$$

Hence, for every bounded measurable h ,

$$\int_{\mathbb{R}} h(u) f_1^{L(T_N)}(u) du = \int_{\mathbb{R}} h(u) u f_0^{L(T_N)}(u) du.$$

Since this equality holds for all bounded measurable functions h , it follows that

$$f_1^{L(T_N)}(u) = u f_0^{L(T_N)}(u)$$

for Lebesgue-almost all $u > 0$. Therefore, for $Y = L(T_N)$, we have $E_0 [I(Y \geq C)Y] = Pr_1(Y \geq C) = Pr_1(T_N \geq C^{T_N})$.

Now, with respect to the modified statement of Proposition 2, for $Y = L(T_N)$, we reconsider

$$\begin{aligned} E_0 I(\Lambda \geq C^\Lambda)Y &= E_0 I(g(M, V) \geq C^\Lambda)L(T_N(M)) \\ &= \iint I(g(u, v) \geq C^\Lambda) L(T_N(u)) f_0^{M,V}(u, v) du dv \\ &= \iint I(g(u, v) \geq C^\Lambda) L(T_N(u)) f_0^M(u) f_0^V(v) du dv \\ &= \iint I(g(u, v) \geq C^\Lambda) L(T_N(u)) f_0^M(u) f_1^V(v) du dv \end{aligned}$$

$$\begin{aligned}
&= \iint I(g(u, v) \geq C^\Lambda) f_1^M(u) f_1^V(v) \, du \, dv \\
&= \iint I(g(u, v) \geq C^\Lambda) f_1^{M,V}(u, v) \, du \, dv \\
&= E_1 I(g(M, V) \geq C^\Lambda) \\
&= \Pr_1(\Lambda \geq C^\Lambda).
\end{aligned}$$

This completes the proof. \square

Proof of Proposition 3

Proof Let T_N be complete under H_k , for $k = 0$ or $k = 1$. It is clear that

$$f_k^V(v) = \int f_k^V(v) f_k^{T_N}(u) \, du, \quad f_k^V(v) = \int f_k^{V|T_N}(v, u) f_k^{T_N}(u) \, du.$$

Thus,

$$0 = \int \left\{ f_k^{V|T_N}(v, u) - f_k^V(v) \right\} f_k^{T_N}(u) \, du = E_k \left\{ f_k^{V|T_N}(v, T_N) - f_k^V(v) \right\},$$

where $f_0^V = f_1^V$. Therefore, $\left\{ f_k^{V|T_N}(v, u) - f_k^V(v) \right\} = 0$ almost surely under H_k , since T_N is complete.

Since T_N is MP, according to Proposition 2.4 of Vexler and Hutson (2024), we have $f_0^{V|T_N} = f_1^{V|T_N}$, which implies

$$\begin{aligned}
0 &= \int \left\{ f_k^{V|T_N}(v, u) - f_k^V(v) \right\} f_k^{T_N}(u) \, du \\
&= E_k \left\{ f_r^{V|T_N}(v, T_N) - f_r^V(v) \right\}, \quad r \in \{0, 1\}, \quad r = k,
\end{aligned}$$

and hence $\left\{ f_r^{V|T_N}(v, u) - f_r^V(v) \right\} = 0$, almost surely.

The proof is complete. \square

Proof of Proposition 4

Proof If X_1 is normally distributed, then T is MP, as shown in Example 4.

Conversely, suppose T is MP. By Proposition 3, T and \mathbf{V} are independent. This independence implies that $E(\bar{X}_n | \mathbf{V} = \mu) = \mu$ and $\text{var}(\bar{X}_n | \mathbf{V} = \sigma^2/n)$. By Theorem 7.2.1 of Bryc (1995), this conditional structure implies that X_1 must be normally distributed. \square

Supplementary information

This supplement presents technical proofs of results stated in the main text which are omitted there for brevity. Section S1 collects remarks that support or extend points made in the main article, including additional results. With respect to the results shown in Sect. 4.1, Section S2 provides a detailed analysis of the Anderson–Darling (AD) and Cramér–von Mises (CvM) tests applied to both the raw observations X_1, \dots, X_n and their absolute values. We examine the asymptotic relative efficiency (ARE) of these tests under a class of symmetric contiguous alternatives. The analysis, based on the framework of Inglot et al. (2000), demonstrates that applying the tests to the absolute values—i.e., folding the data—increases their asymptotic efficiency. Section S3 (“Supplementary to Sect. 4.2”) presents Lemma S3.1 and Proposition S3.1, which develop the RV-based dependence analysis for Hotelling’s T^2 and the trace-normalized T_1^2 statistics defined in Sect. 4.2.

Supplementary Information The online version contains supplementary material available at <https://doi.org/10.1007/s11749-026-01010-1>.

Acknowledgements The authors thank Professor Teresa Ledwina (Institute of Mathematics, Polish Academy of Sciences, Poland) and Professor Chris J. Lloyd (Melbourne Business School, Australia) for insightful and helpful discussions related to the results presented in this work. The authors are grateful to the Editor and the two anonymous referees for their constructive suggestions, which led to a substantial extension and improvement of the results in this paper.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

- Behboodiani J (1990) Some characterization theorems on symmetry. *Comput Stat Data Anal* 10(2):189–192
- Bryc W (1995) The normal distribution: characterizations with applications, vol 100. *Lecture Notes in Statistics*. Springer, New York
- Buehler RJ (1982) Some ancillary statistics and their properties. *J Am Stat Assoc* 77(379):581–589
- Cantoni E, Ronchetti E (2001) Robust inference for generalized linear models. *J Am Stat Assoc* 96(455):1022–1030. <https://doi.org/10.1198/016214501753209004>
- Escoufier Y (1973) Le traitement des variables vectorielles. *Biometrics* 29(4):751–760
- Ferguson TS (1967) On characterizing distributions by properties of order statistics. *Sankhyā: Indian J Stat, Ser A* 29:265–278
- Fraser DAS (2004) Ancillaries and conditional inference. *Stat Sci* 19(2):333–369. <https://doi.org/10.1214/088342304000000323>
- Fujikoshi Y, Himeno T, Wakaki H (2004) Asymptotic results of a high dimensional Manova test and power comparison when the dimension is large compared to the sample size. *J Jpn Stat Soc* 34:19–26
- Ghosh M (2002) Basu’s theorem with applications: a personalistic review. *Sankhyā Indian J Stat* 64:509–531
- Ghosh M, Reid N, Fraser DAS (2010) Ancillary statistics: a review. *Stat Sin* 20:1309–1332
- Govindarajulu Z (1966) Characterization of the exponential and power distributions. *Scand Actuar J* 1966(3–4):132–136

- Hájek J, Šidák Z, Sen PK (1967) *Theory of Rank Tests*. Academic Press, New York
- Hamedani GG, Volkmer H (2003) A characterization of symmetric random variables. *Commun Stat Theory Methods* 32(4):723–728
- Héritier S, Ronchetti E (1994) Robust bounded-influence tests in general parametric models. *J Am Stat Assoc* 89(427):897–904. <https://doi.org/10.1080/01621459.1994.10476822>
- Inglot T, Kallenberg WC, Ledwina T (2000) Vanishing shortcoming and asymptotic relative efficiency. *Ann Stat* 28(1):215–238
- Kagan A (2002) Sufficiency and ancillarity in characterization problems. *J Stat Plan Inference* 102(2):223–228
- Kagan AM, Malinovsky Y (2013) On the Nile problem by Sir Ronald Fisher. *Electron J Stat* 7:1968–1982
- Koning NW, Hemerik J (2024) More efficient exact group invariance testing: using a representative subgroup. *Biometrika* 111(2):441–458. <https://doi.org/10.1093/biomet/asad050>
- Lehmann EL (1950) Some principles of the theory of testing hypotheses. *Ann Math Stat* 21:1–26
- Lehmann EL, Casella G (2006) *Theory of point estimation*. Springer, New York
- Lehmann EL, Scholz F (1992) Ancillarity. *Lect Notes-Monogr Ser* 17:32–51
- Lin C-C, Mudholkar GS (1980) A simple test for normality against asymmetric alternatives. *Biometrika* 67(2):455–461
- Lin GD, Stoyanov JM (2022) New characterizations of the gamma distribution via independence of two statistics by using Anosov’s theorem. *arXiv preprint* [arXiv:2207.11473](https://arxiv.org/abs/2207.11473)
- Mardia KV (1974) Applications of some measures of multivariate skewness and kurtosis in testing normality and robustness studies. *Sankhyā Indian J Stat Ser B* 36(2):115–128
- Pfanzagl J (1968) A characterization of the one parameter exponential family by existence of uniformly most powerful tests. *Sankhyā: Indian J Stat, Ser A*, 30:147–156
- Rosenblatt J, Benjamini Y (2013) Another argument in favour of Wilcoxon’s signed rank test. *arXiv preprint* [arXiv:1311.5354](https://arxiv.org/abs/1311.5354)
- Shiraishi T (1986) Optimum properties of the Wilcoxon signed rank test under a Lehmann alternative. *Tsukuba J Math* 10(1):57–61
- Skovgaard IM (1985) A second-order investigation of asymptotic ancillarity. *Ann Stat* 13(2):534–551
- Spiegelhalter D (1977) A test for normality against symmetric alternatives. *Biometrika* 64(2):415–418
- Toma A, Broniatowski M (2011) Dual divergence estimators and tests: robustness results. *J Multivar Anal* 102(1):20–36. <https://doi.org/10.1016/j.jmva.2010.07.010>
- Vexler A (2020) Univariate likelihood projections and characterizations of the multivariate normal distribution. *J Multivar Anal* 179:104643
- Vexler A, Hutson A (2018) *Statistics in the health sciences: theory, applications, and computing*. CRC Press, New York
- Vexler A, Hutson AD (2024) A characterization of most (more) powerful test statistics with simple non-parametric applications. *Am Stat* 78(1):36–46
- Vexler A, Yu J (2018) To t-test or not to t-test? a p-values-based point of view in the receiver operating characteristic curve framework. *J Comput Biol* 25(6):541–550
- Welsh AH (2011) *Aspects of statistical inference*. Wiley, New York

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.