Linear projections of joint symmetry and independence applied to exact testing treatment effects based on multidimensional outcomes

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Abstract

The growing need for analyzing multivariate aspects of joint data distributions is reinforced by a diversity of experiments based on dependent outcomes. In this sense, different contexts of joint symmetry of data distributions have been dealt with extensively in both theory and practice. Univariate characterizations of properties of multivariate distributions can allow the reduction of the original problem to a substantially simpler one. We focus on research scenarios when vectors $\mathbf{x}$ and $\mathbf{Ax}$ are identically distributed, where $\mathbf{A}$ is a diagonal matrix and absolute values of $\mathbf{A}$'s elements equal to one. It is shown that these scenarios are attractive in new characterizations of joint or mutual independence between random variables. We establish projections of the joint symmetry and independence via the one-dimensional symmetry of linear combinations of $\mathbf{x}$'s components and their interactions. These projections are the most revealing of the multivariate data distribution. The usefulness of the linear projections is exemplified by constructing an efficient nonparametric exact test for joint treatment effects. In this framework, an algorithm for implementing linear projection-based tests is proven. Numerical studies based on generated vectors and a real-data set show that the proposed test can exhibit high and stable power characteristics. The present method can be also used for testing independence between symmetric random vectors.

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1. Introduction

Different aspects of symmetry issues are frequently addressed in both the theoretical and applied statistical literature. For instance, in biostatistics, concepts of symmetry of data distributions play a major role in detecting treatment effects. We consider, for example, the following experiment. In order to evaluate the feasibility and efficacy of a group-based therapy for children with attention-deficit/hyperactivity disorder (ADHD) and severe mood dysregulation (SMD), the study was conducted at the Center for Children and Families, the New York State University at Buffalo. Children ages 7 to 12 with ADHD and SMD were assigned to participate in the experimental 11-week group therapy program. The group-based therapy program consisted of eleven 90-minute sessions with concurrent parent and child groups. Assessments were completed at two time points: Baseline (Week 0) and Endpoint (Week 11). Clinicians measured the continuous Children's Depression Rating Scale–Revised total score (CDRS) and the Young Mania Rating Scale (YMRS) (e.g., Leibenluft et al. [18]). To illustrate research questions of the present paper, we consider data from 10 patients. In this case, the classic paired Wilcoxon signed-rank test based on Baseline- and Endpoint-observations displays p-values of 0.056 and 0.057, when measurements of CDRS and YMRS are used, respectively. However, an effect of the group-based therapy program may be still detected by evaluating a change in joint Baseline and Endpoint distributions of CDRS and YMRS data points (see Section 4). Note that the estimated correlations between CDRS and YMRS are about 0.125 and 0.218 with respect to Baseline and Endpoint, respectively. In this

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example, the relatively small sample size of 10 makes critical concerns regarding: (a) applications of multivariate tests with asymptotically controlled significance levels; (b) applications of multivariate goodness-of-fit pre-tests, and then parametric assumptions (a use of pre-tests can require to adjust outputs of main decision-making procedures).

Paired data based on pre-and post-experimental measurements are frequently encountered in health-related studies. For example, a univariate paired data driven problem can be stated as following: let us first define (Y_{11}, Y_{21}), ..., (Y_{1n}, Y_{2n}) to be a random sample from a bivariate population with absolutely continuous joint distribution function F_{12} and marginal distributions of Y_{1i} and Y_{2i} given as F_1 and F_2, respectively. The general univariate location decision-making problem consists of testing the hypothesis H_0: $\theta = 0$ versus H_1: $\theta \neq 0$ through the relationship $F_1(x) = F_2(x - \theta)$. For example, in biomedical experiments θ may represent the mean or median difference between subject values measured pre-and post-treatment. The paired t-test, the sign test and the Wilcoxon signed-rank test are common statistical procedures for testing H_0: $\theta = 0$ versus H_1: $\theta \neq 0$. These tests are based on the n paired differences $Z_i = Y_{1i} - Y_{2i}$. For instance, the classical nonparametric Wilcoxon signed-rank procedure is a permutation-based method under the assumption that Z is symmetric about zero under the null hypothesis. The biostatistical literature has tended to associate the problem of testing the hypothesis $F_1 = F_2$, e.g., in the context of detecting treatment effects, with the statement of testing the symmetry $\Pr(Z_1 \leq x) = \Pr(-Z_1 \leq x)$ when the forms of the distribution functions F_1 , F_2 and F_{12} are completely unknown (e.g., Wilcoxon [39]).

In modern biostatistical practice, decision-making procedures are often assumed to be based on multiple dependent outcomes. This stimulates the growing need for fundamental developments concerning various aspects of symmetry of multivariate probability distributions in both theory and practice, e.g., [7, 9, 33]. The present paper is intended for focusing on cases where vectors $\mathbf{x}$ and $-\mathbf{x}$ are identically distributed, say $\mathbf{x} \overset{d}{=} -\mathbf{x}$, where the notation $\overset{d}{=}$ means equality in distribution. This statement represents a symmetry of $\mathbf{x}$'s distribution about zero in a natural and common way, e.g., [1]. Our aim is to examine simple univariate characterizations of such form of symmetry, in order to construct a testing strategy for assessing joint symmetry via a known powerful procedure that uses one-dimensional observations.

Nonparametric decision-making evaluations of the vector's distribution symmetry are not simple tasks that can be implemented by directly extending well-known univariate procedures. Perhaps, this is due to the fact that, in the multivariate setting, extensions of null-distribution-free univariate methods, e.g., Kolmogorov-Smirnov or Wilcoxon signed-rank type schemes, are not exact, since their null distributions depend on underlying data distributions. In a theoretical point of view, the critical issue is that the marginal distributions of vector's elements cannot guarantee to depict sufficient information about the corresponding multivariate distribution of the vector (see, e.g., Example 1 in Section 2.1), since a structure of dependence between the vector's components should be examined, whereas this structure is unknown in many cases. Univariate characterizations of properties of the vector's distribution can allow the reduction of the original problem to a substantially simpler one.

We propose to one-to-one map the joint symmetry to the one-dimensional symmetry of linear combinations of $\mathbf{x}$'s components and their interactions. In this context, we refer the reader to Friedman [10] for the explanation: "The most commonly used dimension-reducing transformations are linear projections. This is because they are among the simplest and most interpretable. Moreover, projections are smoothing operations in that structure can be obscured by projection but never enhanced. Any structure seen in a projection is a shadow of an actual (usually sharper) structure in the full dimensionality. In this sense those projections that are the most revealing of the high-dimensional data distribution are those containing the sharpest structure. It is of interest then to pursue such projections."

It turns out that scenarios, when p-dimensional vectors $\mathbf{x}$, $-\mathbf{x}$ and $\mathbf{Ax}$ are identically distributed, where the matrix $\mathbf{A} = \operatorname{diag}\left(h_1, \ldots, h_p\right)$, $|h_i| = 1, i \in \{1, ..., p\}$, $-p < \sum_{i=1}^{p} h_i < p$, can be attractive in new characterizations of joint or mutual independence between random variables. Statistical aspects of independence are very complicated issues in characterizing, evaluations and testing (Lehmann [17]). Note that, in statistical analyses of medical data, exploring joint or mutual dependence among variables is a fundamental issue (e.g., Vexler et al. [34]). For example, it is important to detect the dependence between disease characteristics and a variety of potential predictors to determine significant risk factors. There is substantial literature addressing the development of classes of multivariate distributions in light of independence properties, e.g., [8, 14]. For instance, it is of great interest, to treat distribution families in which zero correlation implies or does not imply independence. In Section 2, we construct characterizations of independence, considering wide families of bivariate and multivariate distributions, including those that have been dealt with extensively in the literature. To this end, we derive univariate linear projections of the multivariate symmetry cases, where vectors $\mathbf{x} \overset{d}{=} -\mathbf{x} \overset{d}{=} \mathbf{Ax}$. This concept can be employed for developing exact tests of independence between random vectors (see Section 5 and the supplementary materials, for several details in this context).

In a case of bivariate symmetry, we refer the reader to Nelsen [22] for an extensive study of concepts and associations related to situations when vectors $\mathbf{x} = [X_1, X_2]^\top$, $-\mathbf{x}$, $[-X_1, X_2]^\top$ and $[X_1, -X_2]^\top$ are identically distributed, where the operator $^\top$ stands for transposition.

Note also that considerations of univariate projections of multivariate statements related to $\mathbf{x} \stackrel{d}{=} -\mathbf{x} \stackrel{d}{=} \mathbf{A}\mathbf{x}$ can be rooted in an interest of the multivariate analysis, in a theoretical context. One of our aims is to collect together some interesting facts and to remind the readers regarding different situations related to the studied issues.

In Section 3, we employ the linear projection related to the statement $\mathbf{x} \stackrel{d}{=} -\mathbf{x}$ in order to conduct an exact density based empirical likelihood ratio (DBELR) procedure for testing multivariate symmetry. The density based empirical likelihood (DBEL) method for developing nonparametric decision-making policies has been addressed in both experimental and theoretical studies, e.g., [11, 21, 31, 38, 40]. Vexler et al. [35] introduced the DBELR test for univariate symmetry. It has been shown that the DBELR approach clearly outperforms known testing procedures, including the classic Wilcoxon signed-rank test, in many cases. In Section 3.1, the DBEL technique is outlined. We then extend the DBELR test to propose a new strategy for detecting joint treatment effects.

Although the use of the linear projection allows to propose a testing mechanism based on univariate combinations of vector-observations' components, we have a complicated issue related to the need for computing the testing procedure over infinitely many linear combinations of vector−observations' components. A neglect of several linear combinations of vector−observations' components in the testing joint symmetry can yield wrong decisions (Hamedani [12]). To overcome this, in Section 3.2, we establish a data−driven algorithm for implementing the proposed DBELR test. This test is exact, i.e. null-distribution-free, for fixed sample sizes. The asymptotic consistency of the proposed nonparametric test is presented.

In Section 3.3, we demonstrate simulation results showing that our test exhibits high and stable power characteristics across a variety of alternatives. In general, in the considered nonparametric framework, there are no most powerful decision-making mechanisms. We select the distance-based weighted decision-making policy recently developed by Chen et al. [7] to be compared with the DBELR test. In contrast to the exact DBELR test, the test of Chen et al. [7] employs a permutation type technique to control the type I error rate approximately that is a common scheme used by many nonparametric testing mechanisms based on multivariate data.

Furthermore, as shown in Section 4, when used to analyze the group-based therapy for children with ADHD and SMD, the proposed test succeeds in finding important treatment effects, illustrating the applicability of the proposed method. Finally, we conclude this paper with a discussion in Section 5. Section 5 discusses how to integrate theoretical propositions shown in the present paper and the decision-making mechanism presented in Section 3 in order to derive tests for independence of random vectors. This approach is experimentally evaluated in the supplementary materials.

2. Linear Characterizations, Symmetry and Independence

In the present section we focus on associations between multivariate symmetric distributions of vectors and univariate distributions of linear combinations of the corresponding vectors' components. We treat a special class of joint symmetries, yielding new characterizations of independence of random variables. It is shown that, in certain situations, the independence notion can be one-to-one mapped to properties of univariate symmetrically distributed linear combinations. Several our research queries are stated in Questions 1-5. The theoretical results are summarized in Propositions 1− 8 that establish univariate projections of joint symmetry and independence. Some proofs of the propositions are included for completeness and contain comments that assist to explain the presented results.

The starting points of our study are associated with Propositions 1 and 2 shown in this section. Proposition 1 uses the Cramér–Wold's concept and can be found in [3]. Proposition 2 directly compiles Proposition 1 and Theorem 2.4 presented in [22]. Up to our knowledge, Propositions 3-8 are new and assist to achieve the following aims. Propositions 3-5 are present in the light of a question regarding situations when concepts of vectors' symmetry could yield independence of vectors' components. For example, in the context of bivariate symmetry, Propositions 3-5 correspond to inverses of the statement "Let X_1 and X_2 be independent. Then different symmetry concepts are equivalent" (see [22], for details). Propositions 7 and 8 depict new principles for deriving univariate projections of the multivariate scenario $\mathbf{x} \stackrel{d}{=} -\mathbf{x} \stackrel{d}{=} \mathbf{A}\mathbf{x}$.

The next subsubsection analyses bivariate cases that are displayed, since: (a) variety statistical topics based on bivariate data have broadened their appeal in recent theoretical and applied studies, e.g., [2, 22]; and (b) the presented analysis is relatively clear, and provides the basic ingredients and explanations for evaluating more general scenarios.

2.1. Bivariate Cases

Let $\mathbf{x} = [X_1, X_2]^\top$ denote a random vector and $Z(a_1, a_2) = a_1X_1 + a_2X_2,\ a_k \in \mathbb{R}^1, k \in \{1, 2\}$, be a linear combination of $\mathbf{x}$'s components. Assume that φ, φ_1 and φ_2 define the characteristic functions of $\mathbf{x}$, X_1 and X_2, respectively. In the present paper we call $\mathbf{x}$ a symmetric random vector if $\varphi(t_1, t_2) = \varphi(-t_1, -t_2)$, for all $t_k \in \mathbb{R}^1, k \in \{1, 2\}$. Certainly, the inversion theorem states that the definition above describes the scenario with $\Pr(X_1 \le u_1, X_2 \le u_2) = \Pr(-X_1 \le u_1, -X_2 \le u_2)$, when $\mathbf{x}$ has a joint density function $f(u_1, u_2) = f(-u_1, -u_2)$, $u_1, u_2 \in \mathbb{R}^1$, e.g., [32, p.56].

It is known that the distribution of $\mathbf{x}$ is uniquely determined by the distributions of linear forms $Z(a_1, a_2),\ a_1, a_2 \in \mathbb{R}^1$ (Cramér–Wold's theorem). Thus, we begin with an analysis of the following question.

Question 1. What can we conclude about the distribution of $\mathbf{x}$, if $Z(a_1, a_2)$ is symmetric, for all $a_k \in \mathbb{R}^1, k \in \{1, 2\}$? Assume $Z(a_1, a_2)$ is symmetric, for all a_1 and $a_2 \in \mathbb{R}^1$. Then

$$\begin{aligned}\varphi(t_1, t_2) = \mathrm{E}\exp\{\mathrm{i}Z(t_1, t_2)\} &= \int \exp(\mathrm{i}u)\, d\Pr\{Z(t_1, t_2) < u\} = \int \exp(\mathrm{i}u)\, d\Pr\{-Z(t_1, t_2) < u\} \\ &= \int \exp(-\mathrm{i}y)\, d\Pr\{Z(t_1, t_2) < y\} = \varphi(-t_1, -t_2),\end{aligned}$$

where $\mathrm{i}^2 = -1$. In this case we note that, since $\varphi(t_1, t_2) = \mathrm{E}\exp(\mathrm{i}t_1X_1 + \mathrm{i}t_2X_2) = \mathrm{E}\cos(t_1X_1 + t_2X_2) + \mathrm{i}\mathrm{E}\sin(t_1X_1 + t_2X_2)$, we have $\mathrm{E}\sin(t_1X_1 + t_2X_2) = 0$ and then

$$\varphi(t_1, t_2) = \mathrm{E}\cos(t_1X_1 + t_2X_2).$$

Assume $\mathbf{x}$ is from a joint density $f(u_1, u_2)$ that satisfies $f(u_1, u_2) = f(-u_1, -u_2)$, for all u_1 and $u_2 \in \mathbb{R}^1$. Then, defining $I(\cdot)$ to be the indicator function, we obtain

$$\begin{aligned}\Pr\{Z(a_1, a_2) < u\} &= \iint I\{a_1u_1 + a_2u_2 < u\} f(u_1, u_2)\, du_1du_2 = \iint I\{a_1u_1 + a_2u_2 < u\} f(-u_1, -u_2)\, du_1du_2 \\ &= \iint I\{-a_1z_1 - a_2z_2 < u\} f(z_1, z_2)\, d(-z_1)\, d(-z_2) = \Pr\{-Z(a_1, a_2) < u\}.\end{aligned}$$

Thus, we conclude with the next result.

Proposition 1. *(Behboodian [3]). The random vector* $\mathbf{x}$ *is symmetric if and only if (iff) any linear combination of the form* $Z(a_1, a_2)$, $a_k \in \mathbb{R}^1, k \in \{1, 2\}$, *is a symmetric random variable.*

Now, let us consider the question below.

Question 2. Is it necessary that $\mathbf{x}$ is symmetric, when X_1 and X_2 are both from symmetric distributions?

Example 1. Consider a case when X_1 and X_2 are symmetric, but $\mathbf{x}$ is not symmetric. Define a symmetric random variable ξ via its characteristic function $\varphi_\xi(t) = (1 - 2|t|)\, I\{|t| \le 0.5\}$. Using the characteristic functions $h_1(t) = (1 - |t|)\, I\{|t| \le 0.5\} + 1/(4|t|)\, I\{|t| > 0.5\}$ and $h_2(t) = (1 - |t|)\, I\{|t| \le 1\}$, we denote a random variable η with the characteristic function $\varphi_\eta(t) = 0.5\exp(\mathrm{i}t)\, h_1(t) + 0.5\exp(\text{-}\mathrm{i}t)\, h_2(t)$. It turns out that we can represent

$$\varphi_\eta(t) = \begin{cases} (1 - |t|)\cos(t), & |t| \le 0.5 \\ \exp(\mathrm{i}t)\, /\, (8|t|) + 0.5\exp(\text{-}\mathrm{i}t)(1 - |t|)\, I\{|t| \le 1\}, & |t| > 0.5, \end{cases}$$

obtaining $\varphi_\eta(t) \ne \varphi_\eta(-t)$ (Stoyanov [30, p. 131]). Assuming ξ and η are independent, we set $X_1 = \xi$ and $X_2 = \xi + \eta$. In this case, we have that X_2 is symmetric with the characteristic function $\varphi_{X_2}(t) = \varphi_\xi(t)\varphi_\eta(t) = (1 - 2|t|)(1 - |t|)\cos(t)\, I\{|t| \le 0.5\}$ (Stoyanov [30, p.131]), as well as

$$\varphi(t_1, t_2) = \mathrm{E}\exp\{\mathrm{i}t_1\xi + \mathrm{i}t_2(\xi + \eta)\} = \varphi_\xi(t_1 + t_2)\, \varphi_\eta(t_2),$$

where $\varphi_\xi(-t_1 - t_2) = \varphi_\xi(t_1 + t_2)$, but $\varphi_\eta(-t_2) \neq \varphi_\eta(t_2)$. Therefore $\mathbf{x}$ is not symmetric. In the context of this example, it is interesting to mention that, according to Burdick [6], there exist independent random variables Y_1 and Y_2 such that Y_1 is symmetric, Y_2 is not symmetric, but $Y_1 + Y_2$ is symmetric. In such cases, defining symmetric random variables $X_1 = Y_1, X_2 = Y_1 + Y_2$, we can see the linear combination $Z(-1, 1)$ is not symmetric and then Proposition 1 says that $= [X_1, X_2]^\top$ is not symmetric.

Example 1 shows that, in general, the statement "X_1 and X_2 are from symmetric distributions" does not guarantee that $\varphi(t_1, t_2) = \mathrm{E}\cos(t_1X_1 + t_2X_2)$.

By virtue of that, for all $a_k \in \mathbb{R}^1, k \in \{1, 2\}$, the linear form $Z(a_1, a_2)$ is symmetric, we have X_1 is symmetric and X_2 is symmetric, considering scenarios with $a_2 = 0$ or $a_1 = 0$, respectively. In this case, it is clear that $\mathrm{E}\sin(t_1X_1) = 0$ and $\mathrm{E}\sin(t_2X_2) = 0$, for all $t_k \in \mathbb{R}^1, k \in \{1, 2\}$. In the context of the joint distribution of X_1 and X_2, the next question can be discussed.

Question 3. Assume $\mathbf{x}$ is symmetric and $\Pr(X_1 \leq u_1, X_2 \leq u_2) = \Pr(-X_1 \leq u_1, X_2 \leq u_2)$. What can we learn from this situation?

Remark 1. Note that we cannot determinate that $\varphi(t_1, t_2) = \varphi(-t_1, -t_2)$ implies, e.g., $\varphi(t_1, t_2) = \varphi(-t_1, t_2)$, in general. For example, if $\mathbf{x}$ has a bivariate normal distribution with $\mathrm{E}\mathbf{x} = \mathbf{0}$ and $\mathrm{E}X_1X_2 \neq 0$, then $\varphi(t_1, t_2) \neq \varphi(-t_1, t_2)$, whereas $\varphi(t_1, t_2) = \varphi(-t_1, -t_2)$.

Now, using the trigonometric product-to-sum identity $\cos(x + y) = \cos(x - y) - 2\sin(x)\sin(y)$, we obtain the following proposition, when $\mathbf{x}$ is symmetric with the characteristic function $\varphi(t_1, t_2) = \mathrm{E}\cos(t_1X_1 + t_2X_2)$.

Proposition 2. *If any linear combination $Z(a_1, a_2)$ is a symmetric random variable and X_1, X_2 are independent, then $\varphi(t_1, t_2) = \varphi(t_1, -t_2)$ and $\varphi(t_1, t_2) = \varphi(-t_1, t_2)$, for all $t_k \in \mathbb{R}^1, k \in \{1, 2\}$. (See also Theorem 2.4 of [22], in this context.)*

Note, for example, that $\varphi(t_1, t_2) = \varphi(t_1, -t_2)$ gives $\Pr(X_1 \leq u_1, X_2 \leq u_2) = \Pr(X_1 \leq u_1, -X_2 \leq u_2)$, $u_k \in \mathbb{R}^1, k \in \{1, 2\}$. The next research query is formulated in the following form.

Question 4. Can we show a reverse of Proposition 2, i.e., the statement: "If any linear combination $Z(a_1, a_2)$ is a symmetric random variable and $\varphi(t_1, t_2) = \varphi(t_1, -t_2)$, then X_1, X_2 are independent"?

Nelsen [22] discussed different concepts of bivariate symmetry, showing that the concepts are equivalent if X_1 and X_2 are independent (Theorem 2.4 of [22]). Question 4 is related to an ability to use the term "iff" in the statement above. Common approaches for characterizing symmetric and jointly symmetric random variables employ concepts of independence, e.g., [7, 33]. In this framework, Question 4 treats a converse statement. Indeed, when $\mathbf{x}$ is from a multivariate normal distribution with zero mean, the functional form of the $\mathbf{x}$'s distribution provides a positive answer to Question 4.

Example 2. Consider the density function related to a mixture of two joint normal distributions with correlations $0 < \rho < 1$ and $-\rho$ in the form:

$$f(x_1, x_2) = 0.5\{g(x_1, x_2, \rho) + g(x_1, x_2, -\rho)\},\ g(x_1, x_2, \rho) = \exp\left\{-0.5\left(x_1^2 - 2\rho x_1x_2 + x_2^2\right)/\left(1 - \rho^2\right)\right\}/\left\{2\pi\left(1 - \rho^2\right)^{1/2}\right\}.$$

This density function was employed by Lancaster [16] in order to illustrate a case, when $\mathbf{x}$ is from $f(x_1, x_2)$, X_1 and X_2 are normally distributed and the correlation of X_1 and X_2 is zero, but X_1, X_2 are dependent. In this scenario, $f(x_1, x_2) = f(-x_1, x_2) = f(x_1, -x_2) = f(-x_1, -x_2)$. This exemplifies that, in general, Question 4 has no simple answer.

Note that, when $\varphi(t_1, t_2) = \mathrm{E}\cos(t_1X_1 + t_2X_2)$ and, e.g., $\varphi(t_1, t_2) = \varphi(t_1, -t_2)$, we have $\mathrm{E}\cos(t_1X_1 + t_2X_2) = \mathrm{E}\cos(t_1X_1 - t_2X_2)$ that implies $\mathrm{E}\sin(t_1X_1)\sin(t_2X_2) = 0$, since $\cos(x + y) = \cos(x - y) - 2\sin(x)\sin(y)$. This leads to

$$\mathrm{E}\cos(t_1X_1 + t_2X_2) = \mathrm{E}\cos(t_1X_1)\cos(t_2X_2),$$

since $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$. We write these observations in the next lemma.

Lemma 1. *Define the claims: (a)* $\varphi(t_1,t_2)=\varphi(t_1,-t_2)$, $\varphi(t_1,t_2)=\varphi(-t_1,t_2)$*; (b)* $\mathrm{E}\sin(t_1X_1)\sin(t_2X_2)=0$*; and (c)* $\varphi(t_1,t_2)=\mathrm{E}\cos(t_1X_1)\cos(t_2X_2)$*. Let any linear combination of the form* $Z(a_1,a_2)$, $a_k\in\mathbb{R}^1, k\in\{1,2\}$*, be a symmetric random variable. Then, (a) implies (b) that implies (c); and (c) implies (b) that implies (a).*

Consider a class of **x**'s distributions with densities that can be expressed as

$$f(x_1,x_2)=r(x_1,x_2)\exp\left(-0.5x_1^2/\sigma_1^2-0.5x_2^2/\sigma_2^2\right),$$

where $r(x_1,x_2)>0$ is a linking function, $\sigma_1^2,\sigma_2^2\in\mathbb{R}^1$ are positive parameters and it is assumed that $r(x_1,x_2)=r(x_1x_2)$, if $r(x_1,x_2)=r(-x_1,-x_2)$. Then, the assumption $\varphi(t_1,t_2)=\varphi(t_1,-t_2)$ implies $r(x_1x_2)=r(-x_1x_2)$ as well as that

$$\begin{aligned}
\mathrm{E}X_1X_2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_1x_2f(x_1,x_2)\,dx_2dx_1=\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{0}x_1x_2f(x_1,x_2)\,dx_2+\int_{0}^{\infty}x_1x_2f(x_1,x_2)\,dx_2\right\}dx_1\\
&= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{0}x_1x_2f(x_1,x_2)\,dx_2+\int_{0}^{\infty}x_1x_2f(x_1,-x_2)\,dx_2\right\}dx_1\\
&= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{0}x_1x_2f(x_1,x_2)\,dx_2-\int_{-\infty}^{0}x_1zf(x_1,z)\,dz\right\}dx_1=0. \qquad (1)
\end{aligned}$$

Thus, in this case, where zero correlation means independence (e.g., Leipnik [19]), we can use Proposition 2 to obtain the following result.

Proposition 3. *Let any linear combination of the form* $Z(a_1,a_2)$, $a_1,a_2\in\mathbb{R}^1$*, be a symmetric random variable and* **x** *be from the density function* $f(x_1,x_2)=r(x_1x_2)\exp\left(-0.5x_1^2/\sigma_1^2-0.5x_2^2/\sigma_2^2\right)$*. Then,* $\varphi(t_1,t_2)=\varphi(t_1,-t_2)$ *or* $\varphi(t_1,t_2)=\varphi(-t_1,t_2)$*, for all* $t_k\in\mathbb{R}^1, k\in\{1,2\}$*, iff* X_1,X_2 *are independent.*

Proposition 3 considers scenarios when zero correlation implies independence. Assume X_1 and X_2 are uncorrelated but not independent. According to Ebrahimi et al. [8], the summand uncorrelated marginal (SUM) bivariate distributions have a great interest in applied and theoretical statistics. In a SUM case, $X_1+X_2\overset{d}{=}X_1'+X_2'$, where the random vector $\mathbf{x}'=\left[X_1',X_2'\right]^\top$ is distributed as $\Pr\left(X_1'\le u_1,X_2'\le u_2\right)=F_1(u_1)F_2(u_2)$ with the marginal distribution functions F_1 and F_2 of X_1 and X_2, respectively. The SUM concept is equivalent to that of sub-independence, where it is assumed that $\varphi(t,t)=\varphi_1(t)\varphi_2(t)$. Following Ebrahimi et al. [8], we study the SUM class of **x**'s distributions with densities in the form

$$f(x_1,x_2)=f_1(x_1)f_2(x_2)\{1+\beta q(x_1,x_2)\},\ \ -q(x_1,x_2)=q(x_2,x_1)=q(-x_1,x_2)=q(x_1,-x_2), \qquad (2)$$

where x_1 and $x_2\in\mathbb{R}^1$, $f_i(x)=dF_i(x)/dx$, $i\in\{1,2\}$, $q(x_1,x_2)$ is a linking function and $\beta\in\mathbb{R}^1$ is a constant that provides $f(x_1,x_2)\ge 0$ and depicts a level of dependence between X_1 and X_2. Supposing that any linear combination $Z(a_1,a_2)$ is a symmetric random variable, we note that X_1 is symmetric and X_2 is symmetric. We then will employ the statement and conditions of Ebrahimi et al. [8]'s Proposition 1 to yield the next result.

Proposition 4. *Let any linear combination of the form* $Z(a_1,a_2)$, $a_k\in\mathbb{R}^1$, $k\in\{1,2\}$*, be a symmetric random variable and* **x** *be from the density function (2). It turns out that the following claims are equivalent: (a)* $\varphi(t_1,t_2)=\varphi(t_1,-t_2)$, $\varphi(t_1,t_2)=\varphi(-t_1,t_2)$*, for all* $t_k\in\mathbb{R}^1, k\in\{1,2\}$*; and (b)* X_1,X_2 *are independent.*

The proof is deferred to the Appendix.

We can remark that density functions satisfied (2) with the marginal densities $f_1(x)=f_2(x)$, $f_1(-x)=f_1(x)$ lead to zero Kendall's tau and zero Spearman's rho (Ebrahimi et al. [8]).

Example 3. To illustrate the family (2), we demonstrate the following scenarios, where X_1 and X_2 are identically $N(0,1)$-distributed:

$$\begin{aligned}
f(x_1,x_2) &= (2\pi)^{-1}\exp\left\{-\left(x_1^2+x_2^2\right)/2\right\}\left[1+\beta x_1x_2\left(x_1^2-x_2^2\right)\exp\left\{-\left(x_1^2+x_2^2\right)/2\right\}\right],\ 0\le\beta\le e^2/4;\\
f(x_1,x_2) &= (2\pi)^{-1}\exp\left\{-\left(x_1^2+x_2^2\right)/2\right\}\left[1+\beta x_1x_2\left(x_1^2-x_2^2\right)\left(x_1^2+x_2^2\right)^{-2}\exp\left\{-\left(x_1^2+x_2^2\right)/2\right\}\right],\ 0\le\beta\le 4.
\end{aligned}$$

It is also interesting to incorporate families of bivariate and multivariate distributions studied by Jogdeo [14] in our research (see also Remark 3, in this context). For example, defining the set of distributions

$$S = \left\{ F : F(x_1, x_2) = \Pr(X_1 \le x_1, X_2 \le x_2) \ \ge \ (\text{or } \le \ ) \ \Pr(X_1 \le x_1) \Pr(X_2 \le x_2), \text{ for all } x_k \in \mathbb{R}^1, k \in \{1, 2\} \right\},$$

we have:

Proposition 5. *Assume that* $\mathbf{x}$*'s distribution is a member of* S *and any linear combination* $Z(a_1, a_2)$, $a_k \in \mathbb{R}^1, k \in \{1, 2\}$, *is a symmetric random variable. Then,* $\varphi(t_1, t_2) = \varphi(t_1, -t_2)$, $\varphi(t_1, t_2) = \varphi(-t_1, t_2)$, *for all* $t_k \in \mathbb{R}^1, k \in \{1, 2\}$, *iff* X_1, X_2 *are independent.*

The proof is deferred to the Appendix.

Note that the class of distribution functions S plays an important role in applications to testing hypotheses of independence, e.g., [14].

In general, intuitively, it seems that, by virtue of Lemma 1 , we can use Taylor's theorem to represent

$$\varphi(t_1, t_2) = \mathrm{E}\cos(t_1 X_1 + t_2 X_2) = \mathrm{E}\cos(t_1 X_1)\cos(t_2 X_2) = \mathrm{E}\left[\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} (-1)^{n+k}(t_1)^{2n}(t_2)^{2k} X_1^{2n} X_2^{2k} / \{(2n)!(2k)!\}\right],$$

when $Z(a_1, a_2)$ is symmetric and $\varphi(t_1, t_2) = \varphi(-t_1, t_2) = \varphi(t_1, -t_2)$. Then, requiring $\mathrm{E}\left(X_1^{2n} X_2^{2k}\right) = \mathrm{E}\left(X_1^{2n}\right)\mathrm{E}\left(X_2^{2k}\right)$, we would obtain a characterization of independence via the symmetry manner, since symmetrically distributed X_1 and X_2 have the characteristic functions $\varphi_1(t) = \mathrm{E}\cos(tX_1)$ and $\varphi_2(t) = \mathrm{E}\cos(tX_2)$, respectively. However, it is known that the equation $\mathrm{E}\left(X_1^n X_2^k\right) = \mathrm{E}\left(X_1^n\right)\mathrm{E}\left(X_2^k\right)$, for all integers $n \ge 0$ and $k \ge 0$, does not guarantee that X_1 and X_2 are independent (Bisgaard and Sasvári [4]). Toward this end, we should state the next proposition in the form below.

Proposition 6. *Assume the following conditions are satisfied:*

(i) *For any real* a_1, a_2, *there exists* $w_a > 0$ *such that* $\mathrm{E}\exp\{w_a |Z(a_1, a_2)|\} < \infty$;

(ii) *any linear combination* $Z(a_1, a_2)$, $a_k \in \mathbb{R}^1, k \in \{1, 2\}$, *is a symmetric random variable.*

Then, for all t_1 *and* $t_2 \in \mathbb{R}^1$ *and integers* $n \ge 0$, $k \ge 0$, *we have* $\varphi(t_1, t_2) = \varphi(t_1, -t_2)$, $\varphi(t_1, t_2) = \varphi(-t_1, t_2)$ *and* $\mathrm{E}\left(X_1^{2n} X_2^{2k}\right) = \mathrm{E}\left(X_1^{2n}\right)\mathrm{E}\left(X_2^{2k}\right)$ *iff* X_1, X_2 *are independent.*

The proof is deferred to the Appendix.

Example 4. We revisit Example 2, where the defined density function satisfies $f(x_1, x_2) = f(-x_1, -x_2)$, $f(x_1, x_2) = f(-x_1, x_2)$ and $f(x_1, x_2) = f(x_1, -x_2)$. It is easy to obtain that $\mathrm{E}\left(X_1^2 X_2^2\right) = 0.5\left\{\left(1 + 2\rho^2\right) + \left(1 + 2(-\rho)^2\right)\right\} \ne \mathrm{E}\left(X_1^2\right)\mathrm{E}\left(X_2^2\right) = 1$ by using a calculation scheme displayed in [2, p. 482]. That is, Proposition 6 does not state that X_1, X_2 are independent.

Remark 2. In order to propose a characterization of symmetry by moment properties, Ushakov [33] applies Condition (i) of Proposition 6 as an essential requirement.

Remark 3. In some classes of $\mathbf{x}$'s distributions, we can significantly simplify Proposition 6's conditions. For example, we can assume that $\mathbf{x}$ is from an infinity divisible distribution. For an extensive review and examples related to infinity divisible distributions, we refer the reader to Bose et al. [5]. In this case, Condition (i) can be eliminated and, by virtue of Pierre [24]'s Theorem 1, we can fix $n = 1, k = 1$ in the equation $\mathrm{E}\left(X_1^{2n} X_2^{2k}\right) = \mathrm{E}\left(X_1^{2n}\right)\mathrm{E}\left(X_2^{2k}\right)$.

The following question is:

Question 5. We have the univariate linear projection of the multivariate statement $f(x_1, x_2) = f(-x_1, -x_2)$. Can we develop univariate projections of the property $f(x_1, x_2) = f(-x_1, x_2) = f(x_1, -x_2)$?

It seems that to investigate Question 5, we need a kind of "second order" linear combinations. To this end, we define the linear combination $V(b_1, b_2, b_3) = b_1 X_1 + b_2 X_2 + b_3 X_1 X_2$ with $b_1, b_2, b_3 \in \mathbb{R}^1$, and present the following result.

Proposition 7. *Assume the conditions below hold:*

(i) *For any real* b_1, b_2, b_3*, there exists* $w_b > 0$ *such that* $\mathrm{E}\exp\{w_b |V(b_1, b_2, b_3)|\} < \infty$*;*

(ii) *any linear combination* $Z(a_1, a_2)$ *with* a_1 *and* $a_2 \in \mathbb{R}^1$*, is a symmetric random variable.*

Then, the following two statements are equivalent:

(iii) *any linear combinations* $V(b_1, 0, b_3)$*,* $V(0, b_2, b_3)$ *are symmetric random variables;*

(iv) $\varphi(t_1, t_2) = \varphi(t_1, -t_2)$*,* $\varphi(t_1, t_2) = \varphi(-t_1, t_2)$*, for all* $t_k \in \mathbb{R}^1, k \in \{1, 2\}$*.*

The proof is deferred to the Appendix.

Note that assertion (iii) of Proposition 7 is somewhat weaker than to require that the vector $[V(b_1, 0, b_3), V(0, b_2, b_3)]^\top$ is from a symmetric distribution.

Remark 4. In a similar manner to testing symmetry showed in Section 3, Propositions 3 - 7 can be employed for assessing a hypothesis of independence with symmetric alternatives (see Section 5, in this context). For example, various studies related to symmetric models' errors, e.g., based on repeated measurements, as well as evaluations of symmetric gambles consider tests for independence of symmetric random variables. Note also that, according to Jogdeo [14], the considered class of distributions S is used in various applications related to testing hypotheses of independence.

2.2. Multivariate Cases

Let $\mathbf{x}$ denote a p-dimensional random vector $\left[X_1, \ldots, X_p\right]^\top$ and $Z(\mathbf{a}) = \mathbf{a}^\top \mathbf{x}$ be a linear combination of $\mathbf{x}$'s components, where a real valued vector $\mathbf{a} = \left[a_1, ..., a_p\right]^\top$. Assume that φ, f are the characteristic and density functions of $\mathbf{x}$, respectively. The distribution of $\mathbf{x}$ is symmetric, $\Pr\left(X_1 \le u_1, ..., X_p \le u_p\right) = \Pr\left(-X_1 \le u_1, ..., -X_p \le u_p\right)$, $u_i \in \mathbb{R}^1, i \in \{1, ..., p\}$, iff φ is symmetric, $\varphi\left(t_1, ..., t_p\right) = \varphi\left(-t_1, ..., -t_p\right)$, $t_i \in \mathbb{R}^1, i \in \{1, ..., p\}$, e.g., [33].

The $p-$dimensional extension of Proposition 1 obviously holds. Regarding the statement of Proposition 2, in the $p-$dimensional case, we can consider various scenarios, for example, (A): $\varphi\left(t_1, t_2, t_3, ..., t_p\right) = \varphi\left(-t_1, ..., -t_m, t_{m+1} ..., t_p\right)$, $1 \le m < p$; or (B): $\varphi\left(t_1, t_2, t_3, ..., t_p\right) = \varphi\left(-t_1, t_2, t_3, ..., t_p\right) = \varphi\left(t_1, -t_2, t_3, ..., t_p\right) = \cdots = \varphi\left(t_1, t_2, ..., t_{p-2}, -t_{p-1}, t_p\right)$. In principle, the mechanisms established in Section 2.1 could be used to study cases defined in a similar manner to (A) and (B). In Section 2.2, we confine our main attention to Scenario (A)−type−fashions, without loss of generality. Scenarios similar to (A) can represent "jointly symmetric" vectors and their "jointly symmetric" components.

Let $Z(\mathbf{a})$ be symmetric, for all $\mathbf{a} \in \mathbb{R}^p$. Then, $X_1, ..., X_p$ are symmetric, $Z([a_1, .., a_m, 0, ..., 0]^\top)$ is symmetric, $\mathrm{E}\sin\left(\sum_{i=1}^m t_i X_i\right) = 0$, and

$$\varphi\left(t_1, ..., t_p\right) = \mathrm{E}\cos\left(\sum\nolimits_{i=1}^p t_i X_i\right) = \mathrm{E}\cos\left(\sum\nolimits_{i=m+1}^p t_i X_i - \sum\nolimits_{i=1}^m t_i X_i\right) - 2\mathrm{E}\sin\left(\sum\nolimits_{i=m+1}^p t_i X_i\right)\sin\left(\sum\nolimits_{i=1}^m t_i X_i\right).$$

Thus, we obtain Scenario (A), if $X_1, ..., X_m$ are independent of $X_{m+1}, ..., X_p$. In a similar manner to the analysis related to Lemma 1, we have $\mathrm{E}\sin\left(\sum_{i=m+1}^p t_i X_i\right)\sin\left(\sum_{i=1}^m t_i X_i\right) = 0$, in (A). Hence, $\varphi\left(t_1, t_2, ..., t_p\right) = \mathrm{E}\left\{\cos\left(\sum_{i=m+1}^p t_i X_i\right)\right.$ $\left.\cos\left(\sum_{i=1}^m t_i X_i\right)\right\}$. In order to derive a $p-$dimensional version of Proposition 6, we assume that, for any real $\mathbf{a} \in \mathbb{R}^p$, there exists $w_a > 0$ such that $\mathrm{E}\exp\{w_a |Z(\mathbf{a})|\} < \infty$, and present

$$\begin{aligned}
\varphi\left(t_1, t_2, t_3, ..., t_p\right) &= \mathrm{E}\left[\sum\nolimits_{k=0}^{\infty}\sum\nolimits_{n=0}^{\infty}(-1)^{n+k}\left(\sum\nolimits_{i=m+1}^p t_i X_i\right)^{2n}\left(\sum\nolimits_{i=1}^m t_i X_i\right)^{2k} / \{(2n)!(2k)!\}\right] \\
&= \mathrm{E}\left[\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(-1)^{n+k}\left\{\sum\nolimits_{r_{m+1}+\cdots+r_p=2n}\prod\nolimits_{i=m+1}^p (t_i X_i)^{r_i}(r_i!)^{-1}\right\}\left\{\sum\nolimits_{r_1+\cdots+r_m=2k}\prod\nolimits_{j=1}^m (t_i X_i)^{r_i}(r_i!)^{-1}\right\}\right],
\end{aligned}$$

where the Multinomial theorem is employed and, e.g., the notation $\sum_{r_{m+1}+\cdots+r_p=2n}$ means the summation is taken over all sequences of integers $r_{m+1}, ..., r_p \ge 0$ such that $r_{m+1} + \cdots + r_p = 2n$. Thus, Scenario A with the requirement $\mathrm{E}\left(\prod_{i=m+1}^p X_i^{r_i}\right)\left(\prod_{j=1}^m X_i^{r_j}\right) = \mathrm{E}\left(\prod_{i=m+1}^p X_i^{r_i}\right)\mathrm{E}\left(\prod_{j=1}^m X_j^{r_j}\right)$, $\sum_{i=m+1}^p r_i{=}2n$, $\sum_{i=1}^m r_i{=}2k$, for all integers n, k leads to $\varphi\left(t_1, ..., t_p\right) = \mathrm{E}\cos\left(\sum_{i=m+1}^p t_i X_i\right)\mathrm{E}\cos\left(\sum_{i=1}^m t_i X_i\right)$ that is to say $X_1, ..., X_m$ are independent of $X_{m+1}, ..., X_p$.

In order to extend Proposition 5, we consider, for example, the set of distributions

$$S = \left\{ F : F(x_1, x_2, x_3) \geq \prod_{i=1}^{3} \Pr(X_i \leq x_i), \text{ for all } x_k \in \mathbb{R}^1, k \in \{1, 2, 3\} \right\}$$

that is the main object of Theorem 3 of Jogdeo [14]. It turns out that, the claim: the triple $\mathbf{x}$ has a distribution from S, $\mathrm{E}|X_i|^3 < \infty, i \in \{1, 2, 3\}$, $\mathrm{E}X_iX_j = \mathrm{E}X_i\mathrm{E}X_j, 1 \leq i \neq j \leq 3$, and $\mathrm{E}X_1X_2X_3 = \mathrm{E}X_1\mathrm{E}X_2\mathrm{E}X_3$ implies the fact: X_1, X_2, X_3 are independent. In the context of Scenario B with $p = 3$, we have, for variables a, b with values of 0 or 1,

$$\begin{aligned} \mathrm{E}X_1X_2^aX_3^b &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2^a x_3^b \int_{-\infty}^{\infty} x_1 f(x_1, x_2, x_3) dx_1 dx_2 dx_3 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2^a x_3^b \left\{ \int_{-\infty}^{0} x_1 f(x_1, x_2, x_3) dx_1 + \int_{0}^{\infty} x_1 f(x_1, x_2, x_3) dx_1 \right\} dx_2 dx_3 = 0, \end{aligned}$$

since $f(x_1, x_2, x_3) = f(-x_1, x_2, x_3)$. This means $\mathrm{E}X_1 = \mathrm{E}X_1X_2 = \mathrm{E}X_1X_3 = \mathrm{E}X_1X_2X_3 = 0$. Similarly, we obtain that $\mathrm{E}X_2 = \mathrm{E}X_2X_3 = 0$. Hence, if we have that $Z(\mathbf{a})$ is from a symmetric distribution, for all $\mathbf{a} \in \mathbb{R}^3$, $\mathbf{x}$'s distribution belongs to S and $\mathrm{E}|X_i|^3 < \infty$, $i = \{1, 2, 3\}$, then $\varphi(t_1, t_2, t_3) = \varphi(-t_1, t_2, t_3) = \varphi(t_1, -t_2, t_3)$ if and only if X_1, X_2, X_3 are independent.

To exemplify a concept for extending Proposition 7, we consider the triple $\mathbf{x}$, Scenario A with $m = 1, p = 3$, the linear combination $V(b_1, ..., b_6) = \sum_{i=1}^{3} b_iX_i + b_4X_1X_2 + b_5X_1X_3 + b_6X_2X_3$ and show the following proposition.

Proposition 8. *Assume the conditions below hold:*

(i) *For any real* $b_i, 1 \leq i \leq 6$, *there exists* $w_b > 0$ *such that* $\mathrm{E}\exp\{w_b |V(b_1, ..., b_6)|\} < \infty$;

(ii) *any linear combination* $Z(\mathbf{a})$ *is a symmetric random variable.*

Then, the following two statements are equivalent:

(iii) *any linear combinations* $V(0, b_2, b_3, b_4, 0, 0)$, $V(b_1, b_2, 0, 0, b_5, 0)$, $V(0, b_2, b_3, 0, b_5, 0)$, $V(b_1, b_2, 0, 0, 0, b_6)$ *are symmetric random variables;*

(iv) $\varphi(t_1, t_2, t_3) = \varphi(-t_1, t_2, t_3)$, $\varphi(t_1, t_2, t_3) = \varphi(t_1, t_2, -t_3)$, *for all* $t_k \in \mathbb{R}^1$, $k \in \{1, 2, 3\}$.

The proof is deferred to the Appendix.

Note that assertion (iii) of Proposition 7 means the vectors $[X_1, X_1X_2]^\top$ and $[X_2, X_1X_2]^\top$ are symmetric, whereas, to consider the trivariate scenario of Proposition 8, we need to address the vectors $[X_2, X_3, X_1X_2]^\top$,..., $[X_1, X_2, X_2X_3]^\top$ in the symmetry context. Assume the triple $\mathbf{x}$ has a distribution from S, satisfying Conditions (i) and (ii) of Proposition 8. In this case, claim (iv) leads to conclude that X_1, X_2, X_3 are independent, in a similar manner to the explanations shown above Proposition 8. We have, for example, that

$$\varphi(t_1, t_2, t_3) = \mathrm{E}\cos(t_1X_1 + t_2X_2 + t_3X_3) = \mathrm{E}\cos(-t_1X_1 + t_2X_2 + t_3X_3) - 2\mathrm{E}\sin(t_1X_1)\sin(t_2X_2 + t_3X_3)$$

with $\mathrm{E}\sin(t_1X_1) = 0$, under Condition (ii). Then, the mutual independence of X_1, X_2, X_3 implies (iv), where, e.g., $\varphi(-t_1, t_2, t_3) = \mathrm{E}\cos(-t_1X_1 + t_2X_2 + t_3X_3)$.

Remark 5. Proposition 8 can yield an extension of the statement of Remark 4 with respect to testing for independence of symmetric random vectors.

3. The DBELR test for treatment effects

In this section, to display an efficient applicability of the linear projections related to joint symmetry, we develop and examine the multivariate DBELR procedure for detecting treatment effects. In Section 3.1, we first briefly introduce the univariate DBEL concept, for completeness sake. In Section 3.2, the multivariate DBELR procedure is developed and its type I error rates and power are analyzed.

3.1. The Univariate Case Revisited

As background relative to the development of the new test we outline the classic empirical likelihood (EL) approach. Assume $Z_1, ..., Z_n$ are independent identically distributed (i.i.d.) univariate random variables with corresponding distribution function F. The classic EL has the distribution function−based form $\prod_{i=1}^n \{F(Z_i) - F(Z_i-)\}$. An empirical estimator of this likelihood is $L_p = \prod_{i=1}^n p_i$, where the components, probability weights, $p_i, i \in \{1, ..., n\}$, maximize L_p, satisfying empirical constraints, e.g., $\sum_{i=1}^n p_i = 1$, $\sum_{i=1}^n p_i Z_i = 0$. Computation of $p_i, i \in \{1, ..., n\}$ is commonly a simple exercise in the use of Lagrange multipliers. The EL technique can be applied to test for hypotheses regarding parameters (e.g., moments) of distributions (e.g., Owen [23], Vexler et al. [37]).

The Neyman−Pearson Lemma states that density−based likelihood ratios can introduce most powerful tests. This motivates nonparametric developments in the "distribution function−based" EL manner, e.g., [21, 38]. The idea is to approximate the likelihood $L_f = \prod_{i=1}^n f(Z_i)$, $f(u) = dF(u)/du$, rewriting L_f as $L_f = \prod_{i=1}^n f_i$, $f_i = f(Z_{(i)})$, where $Z_{(1)} \le Z_{(2)} \le ... \le Z_{(n)}$ are the order statistics based on $Z_1, ..., Z_n$, and $f_1, ..., f_n$ are estimated by maximizing L_f given the empirical constraint related to the rule $\int f(u)du = 1$. This DBEL concept has been successfully used to propose and apply various null−distribution−free testing strategies, e.g., [11, 31, 40].

Consider a problem of testing the null hypothesis H_0: $F = F_0$, $F_0(u) = 1 - F_0(-u)$, for all $-\infty < u < \infty$, against $H_1 : F = F_1, F_1(u) \ne 1 - F_1(-u)$, for some $-\infty < u < \infty$, where F_0 and F_1 are unknown. Let f_1, f_0 denote the density functions of Z under H_1 and H_0, respectively. In this framework, Vexler at al. [35] introduced the following DBELR testing approach. It is shown that the likelihood ratio $\prod_i^n f_1(Z_i)/f_0(Z_i)$ can be approximated by the test statistic

$$V_n(Z_1, ..., Z_n) \quad = \quad \min_{a(n)\le m\le b(n)} \prod_{j=1}^{n} 2m \frac{\{1-(m+1)/(2n)\}}{n\ \Delta_{jm}(Z_1, ..., Z_n)}, \quad a(n) = n^{0.5+\delta},\ b(n) = \min(n^{1-\delta}, n/2),\ \delta \in (0,1/4),$$

where $\Delta_{jm}(Z_1, ..., Z_n) = \sum_{i=1}^n \left\{ I\left(Z_i \le Z_{(j+m)}\right) + I\left(-Z_i \le Z_{(j+m)}\right) - I\left(Z_i \le Z_{(j-m)}\right) - I\left(-Z_i \le Z_{(j-m)}\right) \right\}/2n$ and $Z_{(j)} = Z_{(1)}$, if $j \le 1$, $Z_{(k)} = Z_{(n)}$, if $k \ge n$. The null hypothesis is proposed to be rejected for large values of $V_n(Z_1, ..., Z_n)$.

3.2. The multivariate test

Suppose the researcher observes data points ${}_i\mathbf{x} = \left[X_{1i}, ..., X_{pi}\right]^\top$, $i \in \{1, ..., n\}$, independent realizations of vector $\mathbf{x} = \left[X_1, ..., X_p\right]^\top$, and is interested in testing $H_0 : \mathbf{x} \overset{d}{=} -\mathbf{x}$ versus $H_1 : \mathbf{x} \overset{d}{\ne} -\mathbf{x}$. In order to test for H_0, we may use the statistic $V_n(X_1(\mathbf{u}), ..., X_n(\mathbf{u}))$, where the vector $\mathbf{u} = \left[u_1, ..., u_p\right]^\top \in \mathbb{R}^P$ and the univariate linear combinations $X_i(\mathbf{u}) = \mathbf{u}^\top({}_iX)$, $i \in \{1, ..., n\}$. A concept associated with the Kolmogorov−Smirnov principle for measuring distances between nonparametric hypotheses leads us to propose employing large values of the test statistic

$$TS_n \quad = \quad \max_{\mathbf{u}\in\mathbb{R}^p} V_n(X_1(\mathbf{u}), ..., X_n(\mathbf{u})),$$

for discriminating between H_0 and its alternative hypothesis.

The significant difficulty in computing TS_n is related to an implementation of the maximum of $V_n(X_1(\mathbf{u}), ..., X_n(\mathbf{u}))$ over all values of $u_1, ..., u_p \in \mathbb{R}^1$. We can remark that, e.g., to implement univariate Kolmogorov−Smirnov type statistics, algorithms for conducting maximums employed in the corresponding statistics can be performed by using finite numbers of arguments based on observations. In our case, we directly apply the method introduced by [36]. Toward this end, we recursively define the following system of notations. Let $J_{W,j} = (i_1, j_1, ..., i_{2^{j-1}}, j_{2^{j-1}})$, $J^c_{W,j} = (i_{2^{j-1}+1}, j_{2^{j-1}+1}, ..., i_{2^j}, j_{2^j})$, $J_{U,j} = (i_1, r_1, ..., i_{2^{j-1}}, r_{2^{j-1}})$ and $J^c_{U,j} = (i_{2^{j-1}+1}, r_{2^{j-1}+1}, ..., i_{2^j}, r_{2^j})$ denote integer row−vectors with the components $1 \le i_q \ne r_q \le n$, $1 \le j_q \le n$, where $q \in \{1, ..., 2^j\}$ and $j \in \{1, ..., p-1\}$. We can write $J_{W,1} = (i, j)$ and $J_{U,1} = (i, r)$, for the sake of simplicity. When $J_{W,j} \ne J^c_{W,j}$ and $J_{U,j} \ne J^c_{U,j}$, we denote, for $k \in \{1, ..., p-1\}$, the sets of the random variables $W_k(J_{W,1}) = \left(X_{ki} + X_{kj}\right)/\left(-X_{pj} - X_{pi}\right)$, $U_k(J_{U,1}) = (X_{ki} - X_{kr})/\left(X_{pr} - X_{pi}\right)$, and then, for $j \in \{2, ..., p-1\}$, $k \in \{1, ..., p-j\}$, $W_k\left(J_{W,j}\right) = \left\{W_k\left(J_{W,j-1}\right) - W_k\left(J^c_{W,j-1}\right)\right\}/\left\{W_{p-j+1}\left(J^c_{W,j-1}\right) - W_{p-j+1}\left(J_{W,j-1}\right)\right\}$ and $U_k\left(J_{U,j}\right) = \left\{U_k\left(J_{U,j-1}\right) - U_k\left(J^c_{U,j-1}\right)\right\}/\left\{U_{p-j+1}\left(J^c_{U,j-1}\right) - U_{p-j+1}\left(J_{U,j-1}\right)\right\}$.

In this framework, for example, the notation $W_k(J_{W,1})$ means the sequence

$$(X_{k1} + X_{k1})\left(-X_{p1} - X_{p1}\right)^{-1}, (X_{k1} + X_{k2})\left(-X_{p2} - X_{p1}\right)^{-1}, \ldots, (X_{kn} + X_{km})\left(-X_{pm} - X_{pn}\right)^{-1}.$$

We are now in a position to show the result below.

Table 1: Critical Values of the Proposed Test Statistic, $\log(TS_n)$, defined in Proposition 9.

	Significance level α				Significance level α		
n	0.1	0.05	0.01	n	0.1	0.05	0.01
10	4.073	5.122	7.297	45	4.204	4.817	6.242
15	3.579	4.427	6.758	50	4.257	4.838	6.307
20	3.726	4.463	6.355	55	4.274	4.878	6.337
25	3.850	4.538	6.345	60	4.428	5.056	6.551
30	3.864	4.498	6.179	70	4.631	5.215	6.661
35	3.892	4.486	5.928	80	4.800	5.388	6.782
40	4.066	4.698	6.376	100	5.185	5.769	7.187

Proposition 9. *The test statistic TS_n can be represented in the form*

$$TS_n \quad = \quad \max_{(u_1,\dots,u_p)\in\bigcup_{j=1}^{p} B_j} V_n\left(X_1(\mathbf{u}),\dots,X_n(\mathbf{u})\right),$$

where sets $B_1,\dots,B_p$ contain elements defined via the following algorithms: for $s = 1,\dots,p-1$,

$$B_s \quad = \quad \Bigg[\left(u_1,\dots,u_p\right): \ \textit{if } s>1,\ u_1=\dots=u_{s-1}=0;\ u_s=1;\ \textit{for } d=1,\dots,p-s,\ \textit{given } u_1,\dots,u_{s+d-1},\ \textit{select}$$
$$u_{s+d}\in\left\{\sum_{h=s}^{s+d-1} W_h\left(J_{W,p-d-s+1}\right)u_h\ ,\ \sum_{h=s}^{s+d-1} U_h\left(J_{U,p-d-s+1}\right)u_h\right\}\Bigg] \ \textit{and } B_p=\left\{\left(u_1,\dots,u_p\right): u_1,\dots,u_{p-1}=0, u_p=1\right\}.$$

The proof is deferred to the Appendix.

We refer the reader to [36], for more details, examples and explanations related to the notations used in Proposition 9.

Thus, we propose to reject H_0 as $\log(TS_n) > C_\alpha$, where C_α is an $\alpha-$level test threshold.

In order to control the Type I error of the proposed test we consider the following statement.

Significance levels of the proposed test. It is clear that, the proposed test is exact, the null distribution of the test statistic $\log(TS_n)$ is independent of underlying data H_0-distributions. For example, in Table 1, we tabulate the critical values, C_α, for the proposed test, for different sample sizes and $p = 2$ using an R code (R Development Core Team [25]). The relevant R code is displayed in the supplementary materials. Note that we executed extensive Monte Carlo evaluations that confirmed the robustness of the proposed test with respect to the values of $\delta \in (0, 0.25)$ used in the definition of TS_n. For practical purposes, we suggest $\delta = 0.1$. The DBEL literature confirms that power properties of DBEL type test statistics do not differ substantially for values of $\delta \in (0, 0.25)$, e.g., [31, 38]. To obtain Table 1's results, we derived the Monte Carlo percentiles of the H_0-distribution of the test statistic $\log(TS_n)$ with $\delta = 0.1$ based on 20,000 samples of ${}_i\mathbf{x} = [X_{1i}, X_{2i}]^\top \sim N(0, I_2)$, $i \in \{1,\dots,n\}$, where I_p is a $p-$dimensional identity matrix.

According to the statement below, the proposed test is an asymptotically consistent procedure.

Consistency. To show that the present DBELR procedure is an asymptotic power one test, we use the following notations. Let $\Pr_k$ and E_k denote the probability measure and expectation under $H_k, k = 0, 1$. Denote the density functions $f_k(t;\mathbf{u}) = d\Pr_k\{X_1(\mathbf{u}) \le t\}/dt$, $k \in \{0, 1\}$. Proposition 10 indicates the consistency of the proposed test.

Proposition 10. *Let $\mathbf{x} \in \mathbb{R}^p$ be an absolutely continuously distributed vector. Assume there exists a vector $\mathbf{u}_0 = \left[u_{01},\dots,u_{0p}\right]^\top$ such that the expectations $\mathrm{E}\left[\log\left\{f_k\left(X_1\left(\mathbf{u}_0\right);\mathbf{u}_0\right)\right\}\right]$, $k = 0, 1$, are finite. Then, for a positive threshold $C > 0$, we have $\Pr_1\left\{n^{-1}\log(TS_n) > C\right\} \to 1$, whereas $\Pr_0\left\{n^{-1}\log(TS_n) > C\right\} \to 0$, as $n \to \infty$.*

The proof is deferred to the Appendix.

Remark 6. Corresponding to Remark 4, in order to test for that X_1 and X_2 are independent, we can use the test statistics $TS_{kn} = \max_{\mathbf{u}\in\mathbb{R}^2} V_n\left(Y_{k1}(\mathbf{u}),\dots,Y_{kn}(\mathbf{u})\right)$, $k \in \{1, 2\}$, where $Y_{ki}(\mathbf{u}) = \mathbf{u}^\top\left([X_{ki}, X_{1i}X_{2i}]^\top\right)$, $i \in \{1,\dots,n\}$.

Testing for joint treatment effects. Assume observed vectors ${}_i\mathbf{y} = \left[Y_{1i},\dots,Y_{pi}\right]^\top$ and ${}_i\mathbf{q} = \left[Q_{1i},\dots,Q_{pi}\right]^\top$ display pre- and post-treatment measurements related to individual i, $i \in \{1,\dots,n\}$. In order to detect joint treatment effects

Table 2: The Monte Carlo power of the tests defined in Section 3.2 and [7] ($\alpha = 0.05$).

Design	Test/ n	12	20	50	70	100
A_1	DBELR	0.050	0.048	0.049	0.049	0.049
	C-test	0.043	0.059	0.053	0.051	0.058
A_2	DBELR	0.067	0.060	0.237	0.391	0.632
	C-test	0.054	0.060	0.074	0.081	0.112
A_3	DBELR	0.102	0.187	0.646	0.884	0.994
	C-test	0.083	0.113	0.220	0.338	0.474
A_4	DBELR	0.084	0.166	0.945	0.999	1
	C-test	0.033	0.106	0.109	0.112	0.123
A_5	DBELR	0.124	0.220	0.965	0.999	1
	C-test	0.037	0.099	0.112	0.098	0.112

based on $\{{}_i\mathbf{y}, {}_i\mathbf{q}\}$, $i \in \{1, ..., n\}$, we can perform the procedure below. Following, for example, Section 6 of [13], we first reduce the underlying data to be based on p-dimensional (contrast) observations, ${}_i\mathbf{x} = \left[Q_{1i} - Y_{1i}, ..., Q_{pi} - Y_{pi}\right]^\top$, $i \in \{1, ..., n\}$, and then test the hypothesis that ${}_i\mathbf{x}$, $i \in \{1, ..., n\}$, are symmetrically distributed with center of symmetry 0. In this framework, the Hotelling T^2 test can be employed (Seber [28]), provided that ${}_i\mathbf{x}$, $i \in \{1, ..., n\}$, are from a multivariate normal distribution, whereas the proposed DBELR test can be used in a nonparametric manner.

3.3. Numerical Simulations

We conducted a Monte Carlo study to explore the performance of the proposed testing strategy. In terms of evaluations of nonparametric decision−making procedures, we note that: in the considered framework, *(1)* there are no most powerful tests; and *(2)* it can be assumed that reasonable tests for symmetry based on large samples provide relatively equivalent and powerful outputs.

To judge experimental characteristics of the proposed test, we obtained numerical results executing the recently developed procedure of [7], say C−test. We are grateful for the authors of the paper [7] for providing relevant programming codes to implement their test strategy.

We generated 10,000 independent samples of sizes $n \in \{12, 20, 50, 70, 100\}$ from: (A_1) the bivariate Pearson type VII symmetric distribution (Johnson [15, pp. 117-121]) with parameters $m = 1.7$, $\mu = 0$, and $\Sigma = I$; (A_2) the asymmetrical Clayton copula distribution (R package "*copula*", R Development Core Team [25]) with $N(0, 1)$ −marginal distributions and parameter $\theta = 6$; (A_3) $\mathbf{x} = [X_1, X_2]^\top$, where $X_1 \sim N(0, 1)$, $X_2 \sim Unif(-1, 0.7)$; (A_4) $X_1 = \xi_1 - \exp(10)$, $X_2 \sim Unif(-1, 1)$, where $\xi_1 \sim LN(10, 1)$; (A_5) $X_1 = \exp(10 + \xi_2) - \exp(10)$, $X_2 = \xi_3$, where $[\xi_2, \xi_3]^\top \sim N_2(0, J)$ with covariance matrix $J = \left[\sigma_{ij}\right]$, $1 \le i, j \le 2$, $\sigma_{11} = \sigma_{22} = 1$, $\sigma_{12} = \sigma_{21} = 0.5$.

Table 2 shows the results of the power evaluations of the proposed DBELR test and C−test, when the significance level, α, of the tests for $H_0 : \mathbf{x} \stackrel{d}{=} -\mathbf{x}$ was supposed to be fixed at 5%.

Design A_1 corresponds to H_0 and represents a heavy−tailed symmetric distribution of $\mathbf{x} = [X_1, X_2]^\top$, where random variables X_1 and X_2 are uncorrelated but dependent. In this case, it seems that the permutation type technique for controlling the type I error rate applied in C−test may not work accurately. Table 2 demonstrates that, under A_2−A_5, the DBELR strategy is somewhat more powerful than the distance−based weighted decision-making policy developed by Chen et al. [7]. In A_2, the model of $\mathbf{x}$'s distribution assigns a higher probability to joint extreme negative events than to joint extreme positive events. In this case, the proposed test has approximately a 69%−82% power gain as compared to C-test when $n \ge 50$. Design A_3 is a relatively simple scenario of H_1. Designs A_4, A_5 represent observed vectors with independent/dependent elements from skewed and relatively heavy-tailed distributions. In these cases, the DBELR test provides the power that is about 9 times more than that of C-test, when $n \ge 50$. We can suppose that the C-test is biased under A_4, A_5 with n=12.

Based on the Monte Carlo results, we conclude that the proposed test exhibits high and stable power characteristics under different designs of alternatives.

4. Data Analysis

We briefly demonstrate the present DBELR method for testing with an examination of the group-based therapy for children with ADHD and SMD, as described in Section 1. A total of $n = 17$ children with ADHD and SMD were evaluated during the experimental 11-week group therapy program. Define vectors ${}_i\mathbf{u} = [U_{1i}, U_{2i}]^\top$ and ${}_i\mathbf{v} = [V_{1i}, V_{2i}]^\top$, $i \in \{1, ..., n\}$, to represent the couple [CDRS, YMRS]−observations obtained with respect to Baseline and Endpoint, respectively. The estimated values of $\mathrm{E}\,({}_i\mathbf{u})$, $\mathrm{E}\,({}_i\mathbf{v})$, $\mathrm{var}\,({}_i\mathbf{u})$ and $\mathrm{var}\,({}_i\mathbf{v})$ are $[32.06, 13.71]^\top$, $[25.24, 10.29]^\top$, $\begin{bmatrix} 35.56 & 3.96 \\ 3.96 & 25.72 \end{bmatrix}$ and $\begin{bmatrix} 14.94 & 4.18 \\ 4.18 & 27.22 \end{bmatrix}$, respectively. We performed the proposed test based on ${}_i\mathbf{x} = {}_i\mathbf{u} - {}_i\mathbf{v}$, $i \in \{1, .., n\}$, to detect an effect of the group-based therapy program. The test statistic $\log(TS_n)$ with $\delta = 0.1$ had a value of 11.477 corresponding to p-value= 0.0003. It was observed that Chen et al. [7]'s test (C-test) provided p-value= 0.0017 being coherent with the DBELR test, in this case.

We conducted a bootstrap type data-driven study to examine the proposed test using ideas introduced by Stigler [27]. The considered tests based on the full dataset reject the null hypothesis, indicating a strong evidence of the [CDRS, YMRS]−treatment effects. Thus, it can be assumed that the rejection rate of the null hypothesis for a test based on samples (with relatively small sizes) from the data, ${}_i\mathbf{x}$, $i \in \{1, .., n\}$, can be studied in the context of the efficiency (i.e., the power) of the test. Toward this end, the following procedure was executed. From the full dataset, samples with the sizes 10 and 15 were randomly selected, in order to be tested for $H_0 : \mathbf{x} \overset{d}{=} -\mathbf{x}$ at 5% level of significance. We repeated this strategy 15,000 times calculating the frequencies of the events $\log(TS_{10}) > C_{0.05} = 5.122$ and $\log(TS_{15}) > C_{0.05} = 4.427$. In this framework, the DBELR test did not reject the null hypothesis in 5,175 (the case of n=10) and 473 (n= 15) events, whereas C−test did not reject H_0 in 7,734 and 709 events, respectively. This indicates that our method is more sensitive and reliable as compared to C−test in this data-driven study.

Regarding the dataset used in the example shown in Section 1, we report p−values of 0.033 and 0.045 provided by the DBELR test and C-test, respectively.

Then, the DBELR test can be recommended to be applied as a primary statistical tool in ADHD and SMD studies.

5. Concluding Remarks

For characterizing symmetry of random vectors' distributions, we have proposed using linear combinations of components of the vectors. We have considered the statements: (S1) $\mathbf{x} \overset{d}{=} -\mathbf{x}$, and (S2) $\mathbf{x} \overset{d}{=} -\mathbf{x} \overset{d}{=} \mathbf{A}\mathbf{x}$, where the matrix $\mathbf{A} = \mathrm{diag}\left(h_1, \ldots, h_p\right)$, $|h_i| = 1$, $i \in \{1, ..., p, \}$ $-p < \sum_{i=1}^{p} h_i < p$. Most of our propositions have regarded bivariate symmetry, since: *(a)* bivariate cases play an important role in both theory and practice of statistics; *(b)* the presented analysis is simple, and can be easily extended for more general scenarios; and *(c)* we could not include all diversity of symmetry forms and concepts related to general multidimensional settings. We have involved examples to clarify the explanations of our research points. Propositions 2− 5 have treated a one-to-one mapping between Statement S2, $\mathbf{x} \overset{d}{=} -\mathbf{x} \overset{d}{=} \mathbf{A}\mathbf{x}$, and independence under scenarios when zero correlation implies or does not imply independence. Propositions 1, 7 and 8 have determined forms of linear projections of Statements S1 and S2. The established univariate characterizations of S1 and S2 can allow the reduction of the problem of testing S1 or S2 to a substantially simpler one via employing one-dimensional data points. We have exemplified the usefulness of this approach by developing an efficient nonparametric exact test for joint treatment effects. An algorithm for implementing linear projection-based tests has been proven. In this framework, we have proposed the consistent DBELR test for Statement S1. Its effectiveness in maintaining relatively high power has been demonstrated using simulation and a real data example in bivariate scenarios based on small samples from heavy-tailed distributions.

In the present paper we have touched the problem of characterizing the multivariate independence concept. Note that statistical aspects of independence are very complicated issues in characterizing, evaluations and testing (Lehmann [17]). In future studies, nonparametric tests for independence between random vectors via the proposed characterizations of symmetry are planned to be proposed and extensively examined. In this context, it seems that the DBEL method is an appropriate approach, since its likelihood structure. Thus, according to Remarks 4, 6, the hypothesis: symmetric X_1 and X_2 are independent can be tested exactly using the statistic $Q_n = \log(TS_{1n}) + \log(TS_{2n})$. In the supplementary materials, we show preliminary experimental evaluations of the Q_n-based test. Note that, the DBEL approach and the classical EL method have a similar likelihood-type-rationale. That is, we can combine the

DBEL and EL techniques to develop a test for independence, e.g., using Proposition 6, when the EL approach can test $\mathrm{E}X_1^{2r}X_2^{2\ell} = \mathrm{E}X_1^{2r}\mathrm{E}X_2^{2\ell}$, where $r, \ell \geq 0$. Section 2.2 can provide ingredients to create decision-making procedures for examining independence of random vectors. We think that, in several situations, in order to generalize the statement above in a testing independence context, it is reasonable to transform observed vectors to present realizations from a symmetric multivariate distribution (e.g., [20]). In this case, the present paper can be used for developing general tests for independence based on transformed data points, releasing the symmetry requirement. Further studies are needed to evaluate the proposed approach in these frameworks.

We note also that developments of univariate projections of multivariate statements dealing with $\mathbf{x} \stackrel{d}{=} -\mathbf{x} \stackrel{d}{=} \mathbf{A}\mathbf{x}$ can be of a theoretical interest.

We hope to convince the readers of the benefits of studying different forms of symmetry of multivariate distribution functions and their characterizations applied to creating powerful decision-making mechanisms.

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Appendix. Proofs

Proof of Proposition 4. Assuming Statement (b), we apply Proposition 2 to obtain $\varphi(t_1, t_2) = \varphi(t_1, -t_2)$ and $\varphi(t_1, t_2) = \varphi(-t_1, t_2)$ that is Statement (a). Now, if $\varphi(t_1, t_2) = \varphi(t_1, -t_2)$ and $\varphi(t_1, t_2) = \varphi(-t_1, t_2)$, then, e.g., $f(-x_1, x_2) = f(x_1, x_2)$. In this case, $q(-x_1, x_2) = q(x_1, x_2)$, since $f_1(-x_1) = f_1(x_1)$, when $Z(a_1, a_2)$ is symmetric for all a_1, a_2. However, by (3) we have the condition $q(-x_1, x_2) = -q(x_1, x_2)$. Thus, $q(x_1, x_2) = 0$, i.e., $f(x_1, x_2) = f_1(x_1) f_2(x_2)$ that completes the proof. □

Proof of Proposition 5. If X_1, X_2 are independent, we apply Proposition 2. If $\varphi(t_1, t_2) = \varphi(t_1, -t_2) = \varphi(-t_1, t_2)$, then the proof is implemented by using (1) and the fact that the uncorrelatedness of X_1, X_2 is equivalent to their independence, when $\mathbf{x}$'s distribution is a member of S (see, [14, p. 435]). □

Proof of Proposition 6. It is clear that, if Condition (ii) holds and X_1, X_2 are independent, then Proposition 2 gives $\varphi(t_1, t_2) = \varphi(t_1, -t_2) = \varphi(-t_1, t_2)$. Now, if $\varphi(t_1, t_2) = \varphi(-t_1, t_2) = \varphi(t_1, -t_2)$, in a similar manner to (2), we obtain that $\mathrm{E}\left(X_1^{2n-1}X_2^{2k-1}\right) = \mathrm{E}\left(X_1^{2n-1}\right)\mathrm{E}\left(X_2^{2k-1}\right) = 0$, for all integers $n \geq 1$ and $k \geq 1$. Noting that Condition (i) leads $\varphi, \varphi_1, \varphi_2$ to be analytic characteristic functions (e.g., Ushakov [32]), we apply Theorem 2 of [4] to complete the proof of Proposition 6. □

Proof of Proposition 7. Suppose claim (iii) is satisfied. By virtue of assumption (ii) and Proposition 1, we have that the characteristic function $\varphi(t_1, t_2) = \mathrm{E}\cos(t_1 X_1 + t_2 X_2)$. Since $\cos(x - y) - \cos(x + y) = 2\sin(x)\sin(y)$, we can represent $\varphi(t_1, t_2) = \mathrm{E}\cos(t_1 X_1 - t_2 X_2) - 2\mathrm{E}\sin(t_1 X_1)\sin(t_2 X_2)$. By using Taylor's theorem, we consider $\mathrm{E}\sin(t_1 X_1)\sin(t_2 X_2) = \mathrm{E}\prod_{i=1}^{2}\left(t_i X_i - t_i^3 X_i^3/3! + t_i^5 X_i^5/5! - \cdots\right)$, focusing on $\mathrm{E}(X_1X_2)^{2k+1}X_1^{2\ell}$ and $\mathrm{E}(X_1X_2)^{2k+1}X_2^{2\ell}$ with k and $\ell \in [0, 1, 2, ...]$. Let $f_{121}(u, x_1)$ define a density function of the vector $[(X_1X_2), X_1]^\top$. By virtue of claim (iii) and Proposition 1, $f_{121}(u, x_1) = f_{121}(-u, -x_1)$. Then,

$$\mathrm{E}(X_1X_2)^{2k+1}X_1^{2\ell} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^{2k+1}x_1^{2\ell} f_{121}(u, x_1)\, du dx_1 = \left(\int_{-\infty}^{0}\int_{-\infty}^{0} + \int_{-\infty}^{0}\int_{0}^{\infty} + \int_{0}^{\infty}\int_{-\infty}^{0} + \int_{0}^{\infty}\int_{0}^{\infty}\right) u^{2k+1}x_1^{2\ell} f_{121}(u, x_1)\, du dx_1,$$

where

$$\int_{-\infty}^{0}\int_{-\infty}^{0} u^{2k+1}x_1^{2\ell} f_{121}(u,x_1)\,dudx_1 = \int_{-\infty}^{0}\int_{-\infty}^{0} u^{2k+1}x_1^{2\ell} f_{121}(-u,-x_1)\,dudx_1$$
$$= \int_{\infty}^{0}\int_{\infty}^{0} (-z_1)^{2k+1}(-z_2)^{2\ell} f_{121}(z_1,z_2)\,d(-z_1)\,d(-z_2) = -\int_{0}^{\infty}\int_{0}^{\infty} u^{2k+1}x_1^{2\ell} f_{121}(u,x_1)\,dudx_1$$

as well as

$$\int_{-\infty}^{0}\int_{0}^{\infty} u^{2k+1}x_1^{2\ell} f_{121}(u,x_1)\,dudx_1 = \int_{-\infty}^{0}\int_{0}^{\infty} u^{2k+1}x_1^{2\ell} f_{121}(-u,-x_1)\,dudx_1$$
$$= \int_{\infty}^{0}\int_{0}^{-\infty} (-z_1)^{2k+1}(-z_2)^{2\ell} f_{121}(z_1,z_2)\,d(-z_1)\,d(-z_2) = -\int_{0}^{\infty}\int_{-\infty}^{0} u^{2k+1}x_1^{2\ell} f_{121}(u,x_1)\,dudx_1.$$

Then $\mathrm{E}(X_1X_2)^{2k+1}X_1^{2\ell} = 0$. Similarly, we obtain $\mathrm{E}(X_1X_2)^{2k+1}X_2^{2\ell} = 0$. Therefore, $\mathrm{E}\{\sin(t_1X_1)\ \sin(t_2X_2)\} = 0$. These lead to $\varphi(t_1,t_2) = \mathrm{E}\cos(t_1X_1+t_2X_2) = \mathrm{E}\cos(t_1X_1-t_2X_2) = \varphi(t_1,-t_2)$ and $\varphi(t_1,t_2) = \mathrm{E}\cos(t_1X_1+t_2X_2) = \mathrm{E}\cos(-t_1X_1+t_2X_2) = \varphi(-t_1,t_2)$ that is Statement (iv).

Suppose claim (iv) is satisfied. Then, the distribution

$$\begin{aligned}\Pr\{X_1X_2 \le u_1, X_1 \le x_1\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I\{vg \le u_1, v \le x_1\} f(v,g)\,dvdg = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I\{vg \le u_1, v \le x_1\} f(-v,g)\,dvdg \\ &= \int_{-\infty}^{\infty}\int_{\infty}^{-\infty} I\{-zg \le u_1, -z \le x_1\} f(z,g)\,d(-z)\,dg = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I\{-zg \le u_1, -z \le x_1\} f(z,g)\,dzdg.\end{aligned}$$

That is to say, $\Pr\{X_1X_2 \le u_1, X_1 \le x_1\} = \Pr\{-X_1X_2 \le u_1, -X_1 \le x_1\}$ and $f_{121}(u,x_1) = f_{121}(-u,-x_1)$. This leads to

$$\begin{aligned}\Pr\{V(b_1,0,b_3) \le y\} &= \iint I\{b_1x_1 + b_3u \le y\} f_{121}(u,x_1)\,dudx_1 = \iint I\{b_1x_1 + b_3u \le y\} f_{121}(-u,-x_1)\,dudx_1 \\ &= \int I\{-b_1z_1 - b_3z_2 \le y\} f_{121}(z_2,z_1)\,dz_2dz_1 = \Pr\{-V(b_1,0,b_3) \le y\}.\end{aligned}$$

This way can be easily modified to verify that $\Pr\{V(0,b_2,b_3) \le y\} = \Pr\{-V(0,b_2,b_3) \le y\}$. Thus, claim (iii) is implied by (iv). The proof is complete. □

Proof of Proposition 8. According to the proof strategy of Proposition 7, we first suppose that claim (iii) is satisfied. By assumption (ii), the characteristic function $\varphi(t_1,t_2,t_3) = \mathrm{E}\cos(t_1X_1 + t_2X_2 + t_3X_3)$. We represent

$$\varphi(t_1,t_2,t_3) = \mathrm{E}\cos(t_2X_2 + t_3X_3 - t_1X_1) - 2\mathrm{E}\sin(t_1X_1)\sin(t_2X_2 + t_3X_3).$$

Using Taylor's theorem and the binomial formula, we obtain that

$$\mathrm{E}\sin(t_1X_1)\sin(t_2X_2+t_3X_3) = \mathrm{E}\left\{\sum_{k=0}^{\infty}\frac{(-1)^k(t_1X_1)^{2k+1}}{(2k+1)!}\right\}\left\{\sum_{n=0}^{\infty}(-1)^n\sum_{j=0}^{2n+1}\frac{t_2^jt_3^{2n+1-j}X_2^jX_3^{2n+1-j}}{j!(2n+1-j)!}\right\}.$$

Then, we can focus on $\mathrm{E}X_1^{2k+1}X_2^jX_3^{2n+1-j}$. Assuming, for example, that $0 < j < 2n+1$, we write the following two situations:
(a) $X_1^{2k+1}X_2^jX_3^{2n+1-j} = (X_1X_2)^{2k+1}X_2^{j-2k-1}X_3^{2n+1-j}I(j-2k-1 \ge 0) + (X_1X_2)^jX_1^{2k+1-j}X_3^{2n+1-j}I(2k-j+1 > 0)$, if j is

odd; and

(b) $X_1^{2k+1}X_2^jX_3^{2n+1-j} = (X_1X_3)^{2k+1}X_2^jX_3^{2n-j-2k}I(2n-j-2k\geq 0) + (X_1X_3)^{2n+1-j+1}X_1^{2k-2n+j}X_2^jI(2k-2n+j>0)$, if j is even. In cases (a) and (b), we can find integers $\ell, r > 0$ such that $j = 2\ell+1$ and $j = 2r$, respectively. Then, (a) $X_1^{2k+1}X_2^jX_3^{2n+1-j} = (X_1X_2)^{2k+1}X_2^{2\ell-2k}X_3^{2n-2\ell}I(2\ell-2k\geq 0) + (X_1X_2)^{2\ell+1}X_1^{2k-2\ell}X_3^{2n-2\ell}I(2k-2\ell>0)$, and (b) $X_1^{2k+1}X_2^jX_3^{2n+1-j} = (X_1X_3)^{2k+1}X_2^{2r}X_3^{2n-2r-2k}I(2n-2r-2k\geq 0) + (X_1X_3)^{2n+1-2r+1}X_1^{2k-2n+2r}X_2^jI(2k-2n+2r>0)$, where $2k, 2\ell, 2r, 2n-2\ell, 2n-2r$ are even. By (iii) of the proposition, the vectors $[(X_1X_2), X_2, X_3]^\top$, $[(X_1X_3), X_1, X_2]^\top$ are symmetric. Therefore, $\mathrm{E}X_1^{2k+1}X_2^jX_3^{2n+1-j} = 0$ and, in this manner, we derive $\mathrm{E}\sin(t_1X_1)\sin(t_2X_2+t_3X_3) = 0$ that leads to $\varphi(t_1,t_2,t_3) = \mathrm{E}\cos(t_2X_2+t_3X_3-t_1X_1)$, Similarly, we have $\varphi(t_1,t_2,t_3) = \mathrm{E}\cos(t_1X_1+t_2X_2-t_3X_3)$, corresponding to claim (iv) of the proposition.

Suppose claim (iv) is satisfied. Then, e.g., the characteristic function

$$\begin{aligned}\mathrm{E}\exp\{\mathrm{i}tV(0,b_2,b_3,b_4,0,0)\} &= \int\int\int \exp\{\mathrm{i}t(b_2x_2+b_3x_3+b_4x_1x_3)\}f(x_1,x_2,x_3)\,dx_1dx_2dx_3\\ &= \mathrm{E}\exp\{\mathrm{i}tV(0,-b_2,-b_3,b_4,0,0)\},\end{aligned}$$

since condition (ii) yields that the joint density function $f(x_1,x_2,x_3) = f(-x_1,-x_2,-x_3)$. Then, $f(x_1,x_2,x_3) = f(-x_1,x_2,x_3)$ implies $\mathrm{E}\exp\{\mathrm{i}tV(0,-b_2,-b_3,b_4,0,0)\} = \mathrm{E}\exp\{\mathrm{i}tV(0,-b_2,-b_3,-b_4,0,0)\}$, i.e., $\mathrm{E}\exp\{\mathrm{i}tV(0,b_2,b_3,b_4,0,0)\} = \mathrm{E}\exp\{\mathrm{i}tV(0,-b_2,-b_3,-b_4,0,0)\}$, and $V(0,b_2,b_3,b_4,0,0)$ is symmetric. This scheme can be easily modified to verify that claim (iii) is implied by (iv). The proof is complete. □

Proof of Proposition 9. The proof is similar to that of Proposition 3 in [36] and thus omitted. □

Proof of Proposition 10. To prove Proposition 10, we remark that, following Schuster [26], the concept of the test statistic TS_n development involves the nonparametric estimation $\hat{F}_n(y;\mathbf{u}) = \sum_{i=1}^n \{I(X_i(\mathbf{u})\leq y) + I(-X_i(\mathbf{u})\leq y)\}/2n$ of the symmetric distribution $\mathrm{Pr}_0(X_1(\mathbf{u})\leq y)$. Then, we use the theorem of Dvoretzky, Kiefer and Wolfowitz (Serfling [29, p. 59]), in a similar manner as in the proof scheme of [36]'s Proposition 4. This provides the statement of Proposition 10. □

SUPPLEMENTARY MATERIALS

R Code: Code for Monte Carlo computing the critical values of the null distribution of the proposed test.
Numerical Simulations: Preliminary experimental evaluations of the Q_n-based test that is discussed in Section 5.

References

[1] A. Antille, G. Kersting, W. Zucchini, Testing symmetry, Journal of the American Statistical Association 77 (1982) 639–646.
[2] N. Balakrishnan, C.-D. Lai, Continuous Bivariate Distributions, Springer, New York., 2009.
[3] J. Behboodian, Some characterization theorems on symmetry, Computational statistics and data analysis 10 (1990) 189–192.
[4] T. M. Bisgaard, Z. Sasvári, When does $\mathrm{E}\left(\mathrm{X}^k\mathrm{Y}^\ell\right) = \mathrm{E}\left(\mathrm{X}^k\right)\mathrm{E}\left(\mathrm{Y}^\ell\right)$ imply independence?, Statistics & probability letters 76 (2006) 1111–1116.
[5] A. Bose, A. Dasgupta, H. Rubin, A contemporary review and bibliography of infinitely divisible distributions and processes, Sankhyā: The Indian Journal of Statistics, Series A 64 (2002) 763–819.
[6] D. L. Burdick, A note on symmetric random variables, The Annals of Mathematical statistics 43 (1972) 2039–2040.
[7] F. Chen, S. G. Meintanis, L. Zhu, On some characterizations and multidimensional criteria for testing homogeneity, symmetry and independence, Journal of Multivariate Analysis 173 (2019) 125–144.
[8] N. Ebrahimi, G. G. Hamedani, E. S. Soofi, H. Volkmer, A class of models for uncorrelated random variables, Journal of Multivariate Analysis 101 (2010) 1859–1871.
[9] K.-T. Fang, S. Kotz, K.-W. Ng, Symmetric Multivariate and Related Distributions, Chapman and Hall, New York, 1990.
[10] J. H. Friedman, Exploratory projection pursuit, Journal of the American Statistical Association 82 (1987) 249–266.
[11] N. Guy, O. C. Lancry-Dayan, Y. Pertzon, Not all fixations are created equal: The benefits of using ex-gaussian modeling of fixation durations, Journal of Vision 20:9 (2020) 1–16.
[12] G. G. Hamedani, Nonnormality of linear combinations of normal random variables, The American Statistician 38 (1984) 295–296.
[13] T. P. Hettmansperger, J. Mottonen, H. Oja, Affine-invariant multivariate one-sample signed-rank tests, Journal of the American Statistical Association 92 (1997) 1591–1600.
[14] K. Jogdeo, Characterizations of independence in certain families of bivariate and multivariate distributions, The Annals of Mathematical Statistics 39 (1968) 433–441.
[15] M. E. Johnson, Multivariate Statistical Simulation, Wiley: New York, 1987.
[16] H. O. Lancaster, Zero correlation and independence, Australian Journal of Statistics 1 (1959) 53–56.

[17] E. L. Lehmann, Nonparametrics: Statistical Methods Based on Ranks, Oakland, CA: Holden-Day, 1975.
[18] E. Leibenluft, D. Chaney, K. E. Towbin, R. K. Bhangoo, D. S. Pine, Defining clinical of juvenile mania, American Journal of Psychiatry 160 (2003) 430–437.
[19] R. Leipnik, When does zero correlation imply independence?, The American mathematical monthly 68 (1961) 563–565.
[20] N. Loperfido, Linear transformations to symmetry, Journal of Multivariate Analysis 129 (2014) 186–192.
[21] A. K. Nanda, S. Chowdhury, Shannon's entropy and its generalisations towards statistical inference in last seven decades, International Statistical Review (2021) In Press.
[22] R. B. Nelsen, Some concepts of bivariate symmetry, Journal of Nonparametric Statistics 3:1 (1993) 95–101.
[23] A. B. Owen, Empirical Likelihood, Chapman and Hall/CRC, New York, 2001.
[24] P. A. Pierre, Infinitely divisible distributions, conditions for independence, and central limit theorems, Journal of Mathematical Analysis 33 (1971) 341–354.
[25] R Development Core Team, R: A Language and Environment for Statistical Computing., R Foundation for Statistical Computing, Vienna, Austria. ISBN 3-900051-07-0 ¡http://www.R-project.org¿., 2012.
[26] E. F. Schuster, Estimating the distribution function of a symmetric distribution, Biometrika 62 (1975) 631–635.
[27] E. F. Schuster, Do robust estimators work with real data?, The Annals of Statistics 5 (1977) 1055–1098.
[28] G. A. F. Seber, Multivariate Observations, Wiley: New York, 1984.
[29] R. J. Serfling, Approximation Theorems of Mathematical Statistics, Wiley: New York, 2009.
[30] J. M. Stoyanov, Counterexamples in Probability: Third Edition, Dover Publications. New York, 2014.
[31] W.-M. Tsai, A. Vexler, G. Gurevich, An extensive power evaluation of a novel two-sample density-based empirical likelihood ratio test for paired data with an application to a treatment study of attention-deficit/hyperactivity disorder and severe mood dysregulation, Journal of Applied Statistics 40 (2013) 1189–1208.
[32] N. G. Ushakov, Selected Topics in Characteristic Functions, VSP, Utrecht., 1999.
[33] N. G. Ushakov, One characterization of symmetry, Statistics and Probability Letters 81 (2011) 614–617.
[34] A. Vexler, X. Chen, A. D. Hutson, Dependence and independence: Structure and inference, Statistical Methods in Medical Research 26 (2017) 2114–2132.
[35] A. Vexler, G. Gurevich, A. D. Hutson, An exact density-based empirical likelihood ratio test for paired data, Journal of Statistical Planning and Inference 143 (2013) 334–345.
[36] A. Vexler, G. Gurevich, L. Zou, Exact multivariate two-sample density-based empirical likelihood ratio tests applicable to retrospective and group sequential studies, Preprint arXiv:2101.04805 (2021).
[37] A. Vexler, G. Tao, A. D. Hutson, Posterior expectation based on empirical likelihoods, Biometrika 101 (2014) 711–718.
[38] A. Vexler, W.-M. Tsai, A. D. Hutson, A simple density-based empirical likelihood ratio test for independence, The American Statistician 68 (2014) 158–169.
[39] F. Wilcoxon, Individual comparisons by ranking methods, Biometrics 1 (1945) 80–83.
[40] L. Zou, A. Vexler, J. Yu, H. Wan, A sequential density-based empirical likelihood ratio test for treatment effects, Statistics in Medicine 38 (2019) 2115–2125.

Linear Projections of Joint Symmetry and Independence Applied to Exact Testing Treatment Effects Based on Multidimensional Outcomes

Supplementary Material

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1. **##**
   ####### R code to calculate the critical values of the proposed test
   ##

```
library(MASS)
#Sample size
n1<-35
delta<-0.1
Test_Stat1<- array()
####### DBELR test statistic function #######################
DBEL_test1<- function(cc){
  txx1<- t(cc)%*%t(XX1)  #linear combination (cc) of data XX1
  x1<- txx1[1:n1]
  sx<-sort(x1)
  ###############################################
  #######obtain the test statistic############
  ###############################################
  m<-c(round(n1^(delta+0.5)):min(c(round((n1)^(1-delta)),round(n1/2))))
  a<-replicate(n1,m)
  rm<-as.vector(t(a))
  L<-c(1:n1)- rm
  LL<-replace(L, L <= 0, 1 )
  U<-c(1:n1)+ rm
  UU<-replace(U, U > n1, n1)
  xL<-sx[LL]
```

```
  xU<-sx[UU]
  F<-ecdf(-sx)(xU)+ecdf(sx)(xU)-(ecdf(-sx)(xL)+ecdf(sx)(xL))
  F<-0.5*F
  F[F==0]<-1/(n1)
  I<-2*rm*(1-(rm+1)/(2*n1))/( n1*F )
  ux<-array(I, c(n1,length(m)))
  tstat1<- log(min(apply(ux,2,prod)))
  Test_Stat<- tstat1
  return(Test_Stat)
}
##################################################################
#The total number of Monte Carlo generations
MC<-5000
for(mc in 1:MC){
  #Simulate bivariate normal random sample
  Sigma <- matrix(c(1,0,0,1),2,2)
  XX1<-mvrnorm(n = n1, rep(0, 2), Sigma)
  XX2<-(-XX1)
  Ws<-c()
  XXX1<- XX1
  XXX2<- XX2
  XXX12<-rbind(XXX1,XXX2)

  #Compute linear combinations
  for (i in 1:(length(XXX12[,1])-1)) {
    for (j in (i+1) : length(XXX12[,1])) {
      lz1<- XXX12[i,]
      lz2<- XXX12[j,]
      a2<- (lz1[1]-lz2[1])/(lz2[2]-lz1[2])
      Ws<- c(Ws,a2)
    }
  }

  Ws<- unique(Ws)
 Ws<-Ws[Ws!='Inf']
  Ws<-Ws[Ws!='-Inf']
  Ws<-Ws[Ws!='NaN']
  Ws<- c(Ws,1,0)
  lz_a1<- c(rep(1,length(Ws)-2),0,1)
  xy_lab<-cbind(lz_a1,Ws)
  #Find the maximum test statistic
  stat<<-apply(xy_lab,1,DBEL_test1)
  Test_Stat1[mc]<-max(stat)

  #Track the number of iterations
  print(c('mc=',mc))
```

```
 #print the simulated 0.95 quantile
 print(quantile(Test_Stat1,0.95))
}
```

Note that, in order to generate vectors under designs A_1 and A_2 applied in Section 3.3, the following codes can be used.

```
####A1: Pearson VII Alternative######
library(MASS)
m<-1.7
zz<-mvrnorm(n1, c(0,0), diag(x = 1, 2,2))
df<-2*m-2
ss<-rchisq(n,df,ncp=0)
Y<-df/(sqrt(ss))*zz
XX1<- cbind(Y[,1],Y[,2])

############A2
library(pacman)
p_load(copula,car)
my.cop<- archmCopula(family = "clayton", dim=2, param = 6)
 #Choose marginal distribution and set their parameters
 my.margins = c("norm", "norm")
 my.parms = list(list(mean=0, sd=1), list(mean=0, sd=1))
 #Integrate them in one model
 myBvd = mvdc(copula=my.cop, margins= my.margins, paramMargins = my.parms)
 #Draw n1 random sample
 myBvd.sim<- rMvdc(n1, myBvd)
 XX1<- myBvd.sim
```

Remark S1. Extensive Monte Carlo evaluations based on realizations of $\mathbf{X}$ with a variety of sample sizes *n* showed that multiple uses of the R function '*optim*' (R Development Core Team, 2012) executed with initial values equating to different empirical quantiles of $\left(u_1,...,u_p\right) \in B_k$, $k = 1,..., p,$ can significantly reduce the computation time of the proposed procedure, when $p > 2$.

2. Numerical Simulations: Preliminary experimental evaluations of the Q_n-based test that is discussed in Section 5.

We carried out a Monte Carlo study to evaluate the performance of the Q_n-based test, where $Q_n = \log(TS_{1n}) + \log(TS_{2n})$ with $TS_{kn}, k = 1, 2,$ that are defined in Remark 6. The R code presented above can be easily modified to implement the Q_n-based test. In terms of evaluations of nonparametric decision-making procedures, we note that: in the considered framework, there are no most powerful tests.

The Q_n-based test is exact, meaning that it is distribution-free under H_0, i.e. the H_0-distribution of Q_n does not depend on underlying distributions of symmetric and independent X_1 and X_2. Thus, the critical values for the Q_n-based test can be accurately approximated using Monte Carlo techniques. The generated values (10,000 replications) of the test statistic Q_n were used to determine the critical values $C_{\alpha=0.05}$ of the null distribution of Q_n at the significance level $\alpha = 0.05$: $C_{\alpha=0.05} = 17.160$, if $n = 20$; $C_{\alpha=0.05} = 22.407$, if $n = 50$; $C_{\alpha=0.05} = 26.510$, if $n = 75$.

In this study we attended to the following form of dependence:

$$F:\ X_1 \sim N(0,1), X_2 = \Phi\left(X_1^2\right) - \Phi\left(\xi^2\right), \xi \sim N(0,1), \Phi\left(x\right) = \int_{-\infty}^{x} \exp(-z^2/2)dz / \left(2\pi\right)^{1/2}.$$

It is interesting to depict the following histogram related to the X_2 distribution based on 100000 generations of X_2.

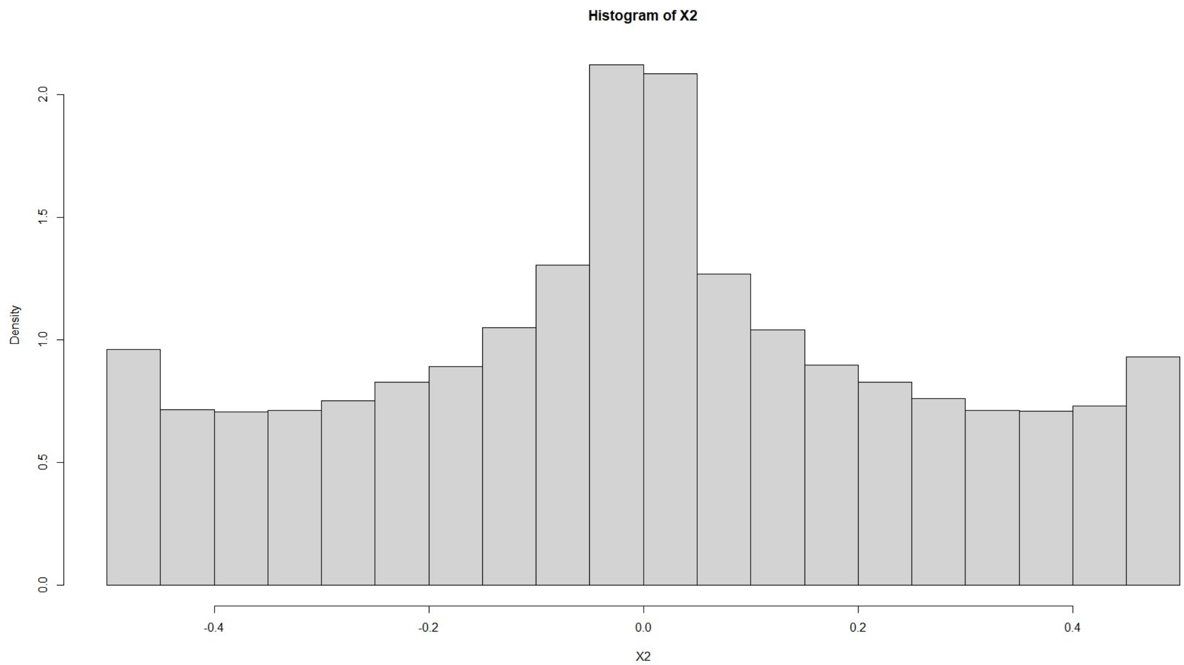

Table S1 shows the results of the power evaluations of the Q_n-based test ("Q_n") and the classical tests ("Pearson", "Kendall") via the Monte Carlo study based on 10,000 replications of $\mathbf{X}_1, \ldots, \mathbf{X}_n$ for the design corresponding to F at each sample size n. This study demonstrates the Q_n-based test is superior to the considered classical tests in the scenarios under the F-design. The power differences between the Q_n-based test and the classical tests become more substantial as the sample size increases.

Table S1. *The Monte Carlo power of the tests* $(\alpha = 5\%)$.

Tests	Design F		
	Sample size (n)		
	20	50	70
Q_n	0.09592326	0.400	0.6943852

Pearson	0.09961262	0.105	0.09452736
Kendall	0.09610773	0.124495	0.1179815

Reference

[1] R Development Core Team. (2012), *R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing.* Vienna, Austria. ISBN 3-900051-07-0 <http://www.R-project.org>.